Let an infinite sequence of $a_i \in \mathbb{R}$ satisfy $1 = a_1 \geq a_2 \geq a_3 \geq \cdots$ and $a_n = a_{2n} + a_{2n+1}$ for all possible n. Show that the following holds for $r = 2026^{2026}$.

$$\frac{1}{r} \leq a_r \leq \frac{2}{r+1}$$

Enoch Yu

$\frac{1}{n} \leqslant a_n \leqslant \frac{2}{n+1}$

$a_1 = a_2 + a_3$

$a_2 = a_4 + a_5$

$a_3 = a_6 + a_7$

lemma $a_n + \cdots + a_{2n-1} = 1$

$k+1$ $\quad a_8 \; a_9 \; a_{10} \; a_{11} \; a_{12} \; a_{13} \mid a_{14} \; a_{15}$

k $\quad a_4 \; a_5 \; a_6 \mid a_7$

$a_2 \quad a_3$

$a_1 = 1$ $\square$

$\frac{1}{n} \leq a_n$

FTSOC, let $a_n < \frac{1}{n}$

$a_n, \ldots, a_{2n-1} < \frac{1}{n}$

$\sum_{i=n}^{2n-1} < \frac{1}{n} \cdot n = 1$

$a_n \leq \frac{2}{n+1}$

$a_{2n-1} \leq \frac{1}{n}$? $\Rightarrow$ true

FTSOC, $a_{2n-1} > \frac{1}{n}$

$\sum_{i=n}^{2n-1} \geq \frac{1}{n} \cdot n = 1$

$a_{2n-1} \leq \frac{1}{n} = \frac{2}{2n-1+1}$

For odd n, $a_n \leq \frac{2}{n+1}$

$a_{2n-2} \leq a_{2n-1} \leq \frac{1}{n}$

$a_{2n-2} \leq \frac{1}{n} \Rightarrow a_{2n} \leq \frac{1}{n+1}$

$a_{2n+1} \leq \frac{2}{2n+1+1} = \frac{1}{n+1}$

$a_n = a_{2n} + a_{2n+1}$

$a_{2n} + a_{2n+1} \leq \frac{1}{n+1} + \frac{1}{n+1}$

$a_n \leq \frac{2}{n+1}$

$\square$