7.29.2025 AMC 10 Free Class Preview Homework Solution

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Problem 1.

Solve for real x, y, and z for the system of equations

$$\begin{cases} x^2 + y^2 + z^2 = \frac{9}{4} \\ -8x + 6y - 24z = 39 \end{cases}$$

Solution Because x, y, and z are real numbers, the two equations could be exploited.

$$x^2 + y^2 + z^2 = \frac{9}{4}$$
$$-4x + 3y - 12z = \frac{39}{2}$$
$$\left(x^2 + 4x + 4\right) + \left(y^2 - 3y + \frac{9}{4}\right) + \left(z^2 + 12z + 36\right) = \frac{9}{4} - \frac{39}{2} + 4 + \frac{9}{4} + 36$$
$$(x+2)^2 + \left(y - \frac{3}{2}\right)^2 + (z+6)^2 = 25$$

There are infinitely many solutions. □

Problem 2.

Find integer solutions to the following equation.

$$x\sqrt{2 - y^2} + y\sqrt{2 - x^2} = 2$$

Solution Notice that both xy and y can either be $-1, 0, 1$. Moreover, the equation is symmetric. The only possible solution is $\boxed{(1,1)}$. □

Problem 3.

Suppose x and y are real numbers. Find the minimum values of

$$u = x^2 + 2xy + 5y^2 + 2x - 6y + 7.$$

Solution Because there exists x^2, y^2, xy, x, y, it is natural to immediately think of the factorization form $(x \pm y + C)^2$.

$$(x + y + 1)^2 + 4y^2 - 8y + 6 = 0$$
$$(x + y + 1)^2 + 4(y - 1)^2 + 2 = 0$$

Thus, the minimum value of u is $\boxed{2}$. □

Problem 4.

Find the maximum value of

$$y = 2x - 5 + \sqrt{15 - 4x}$$

for $x \in \mathbb{R}$.

Solution Let $X = \sqrt{15 - 4x}$. Therefore, the maximum value of $-\frac{X^2}{2} + X + \frac{5}{2}$ must be found.

$$\begin{aligned} -\frac{X^2}{2} + X + \frac{5}{2} &= -\frac{1}{2}\left(X^2 - 2X + 1\right) + 3 \\ &= -\frac{1}{2}(X-1)^2 + 3 \end{aligned}$$

Because X is a positive number, the maximum value of y is $\boxed{3}$. □

Problem 5.

Suppose $\frac{1}{2} \leq x^2 + 4y^2 \leq 2$ is true for real numbers x and y.

1. Find the greatest value of $x^2 - 2xy + 4y^2$
2. Find the smallest value of $x^2 - 2xy + 4y^2$

Part 1.

Solution I. Completing square method may provide the maximum value.

$$\begin{aligned} x^2 + 4y^2 &= \frac{2}{3}\left(x^2 - 2xy + 4y^2\right) + \frac{1}{3}\left(x^2 + 4xy + 4y^2\right) \\ &= \frac{2}{3}\left(x^2 - 2xy + 4y^2\right) + \frac{1}{3}(x + 2y)^2 \end{aligned}$$

Continuing,

$$\frac{1}{2} \leq \frac{2}{3}\left(x^2 - 2xy + 4y^2\right) + \frac{1}{3}(x + 2y)^2 \leq 2$$

Notice that when $x = -2y$, $x^2 - 2xy + 4y^2 \leq 3$. However, other cases will lead to $x^2 - 2xy + 4y^2$ being less than or equal to a value smaller than 3. Therefore, $x^2 - 2xy + 4y^2$ will reach the maximum 3 when $x = -2y$, which is the case that provides the maximum value. □

Solution II. Using AM-GM inequality, it is evident that $x^2 + 4y^2 \geq |4xy|$. In other words, $|2xy| \leq 1$. Moreover,

$$\frac{1}{2} - 2xy \leq x^2 - 2xy + 4y^2 \leq 2 - 2xy$$

Consider the case when $x = -1$ and $y = \frac{1}{2}$. $x^2 + 4y^2$ will reach its maximum while xy will face its minimum. Thus, $2 + 1$, or $\boxed{3}$ is the maximum value. □

Part 2.

Solution Completing square method could be used.

$$\begin{aligned} x^2 + 4y^2 &= 2\left(x^2 - 2xy + 4y^2\right) - \left(x^2 - 4xy + 4y^2\right) \\ &= 2\left(x^2 - 2xy + 4y^2\right) - (x - 2y)^2 \end{aligned}$$

Therefore,

$$\frac{1}{2} \leq 2\left(x^2 - 2xy + 4y^2\right) - (x - 2y)^2 \leq 2$$

Notice that $x^2 - 2xy + 4y^2$ will reach its minimum *iff* $x = 2y$. Thus, the minimum value of $x^2 - 2xy + 4y^2$ is $\boxed{\frac{1}{4}}$. □