7.29.2025 AMC 10,12 Free Class Preview Homework Solution

Enoch Yu

July 2025

Problem 1.

Suppose x and y are real numbers. Find the minimum values of

$$u = x^2 + 2xy + 5y^2 + 2x - 6y + 7.$$

Solution Because there exists x^2, y^2, xy, x, y, it is natural to immediately think of the factorization form $(x \pm y + C)^2$.

$$(x + y + 1)^2 + 4y^2 - 8y + 6 = 0$$
$$(x + y + 1)^2 + 4(y - 1)^2 + 2 = 0$$

Thus, the minimum value of u is $\boxed{2}$. $\square$

Problem 2.

Find the maximum value of

$$y = 2x - 5 + \sqrt{15 - 4x}$$

for $x \in \mathbb{R}$.

Solution Let $X = \sqrt{15 - 4x}$. Therefore, the maximum value of $-\frac{X^2}{2} + X + \frac{5}{2}$ must be found.

$$\begin{aligned} -\frac{X^2}{2} + X + \frac{5}{2} &= -\frac{1}{2}\left(X^2 - 2X + 1\right) + 3 \\ &= -\frac{1}{2}\left(X - 1\right)^2 + 3 \end{aligned}$$

Because X is a positive number, the maximum value of y is $\boxed{3}$. $\square$

Problem 3.

Suppose $\frac{1}{2} \leq x^2 + 4y^2 \leq 2$ is true for real numbers x and y.

1. Find the greatest value of $x^2 - 2xy + 4y^2$
2. Find the smallest value of $x^2 - 2xy + 4y^2$

Part 1.

Solution I. Completing square method may provide the maximum value.

$$\begin{aligned} x^2+4y^2 &= \frac{2}{3}\left(x^2-2xy+4y^2\right)+\frac{1}{3}\left(x^2+4xy+4y^2\right) \\ &= \frac{2}{3}\left(x^2-2xy+4y^2\right)+\frac{1}{3}(x+2y)^2 \end{aligned}$$

Continuing,

$$\frac{1}{2} \leq \frac{2}{3}\left(x^2-2xy+4y^2\right)+\frac{1}{3}(x+2y)^2 \leq 2$$

Notice that when $x=-2y$, $x^2-2xy+4y^2 \leq 3$. However, other cases will lead to $x^2-2xy+4y^2$ being less than or equal to a value smaller than 3. Therefore, $x^2-2xy+4y^2$ will reach the maximum 3 when $x=-2y$, which is the case that provides the maximum value. □

Solution II. Using AM-GM inequality, it is evident that $x^2+4y^2 \geq |4xy|$. In other words, $|2xy| \leq 1$. Moreover,

$$\frac{1}{2}-2xy \leq x^2-2xy+4y^2 \leq 2-2xy$$

Consider the case when $x=-1$ and $y=\frac{1}{2}$. x^2+4y^2 will reach its maximum while xy will face its minimum. Thus, $2+1$, or $\boxed{3}$ is the maximum value. □

Part 2.

Solution Completing square method could be used.

$$\begin{aligned} x^2+4y^2 &= 2\left(x^2-2xy+4y^2\right)-\left(x^2-4xy+4y^2\right) \\ &= 2\left(x^2-2xy+4y^2\right)-(x-2y)^2 \end{aligned}$$

Therefore,

$$\frac{1}{2} \leq 2\left(x^2-2xy+4y^2\right)-(x-2y)^2 \leq 2$$

Notice that $x^2-2xy+4y^2$ will reach its minimum *iff* $x=2y$. Thus, the minimum value of $x^2-2xy+4y^2$ is $\boxed{\frac{1}{4}}$. □

Problem 4.

Find

$$\frac{1}{s_{\max}}+\frac{1}{s_{\min}}$$

if $s=x^2+y^2$ with real x and y satisfying

$$4x^2-5xy+4y^2=5.$$

Solution Using quadratic formula, x could be written in terms of y.

$$x=\frac{5y \pm \sqrt{80-39y^2}}{8}$$

Therefore, $-\sqrt{\frac{80}{39}} \le x \le \frac{80}{39}$ and $-\sqrt{\frac{80}{39}} \le y \le \frac{80}{39}$ since the equation is symmetric.

$$\begin{aligned} s &= \left(\frac{5y \pm \sqrt{80-39y^2}}{8}\right)^2 + y^2 \\ &= \frac{25y^2 + 80 - 39y^2 \pm 10y\sqrt{80-39y^2} + 64y^2}{64} \\ &= \frac{50y^2 \pm 10y\sqrt{80-39y^2} + 80}{64} \end{aligned}$$

Using the range for y, the following inequalities could be derived.

$$\begin{aligned} 0 \le \sqrt{80-39y^2} &\le \sqrt{80} \\ 0 \le 10y\sqrt{80-39y^2} &\le 10\sqrt{\frac{80}{39}} \cdot \sqrt{80} \\ 80 \le 50y^2 + 10y\sqrt{80-39y^2} + 80 &\le \frac{800}{\sqrt{39}} + \frac{400}{39} + 80 \\ \frac{80}{64} \le \frac{50y^2 + 10y\sqrt{80-39y^2} + 80}{64} &\le \frac{\frac{800}{\sqrt{39}} + \frac{400}{39} + 80}{64} \end{aligned}$$

Similarly,

$$\begin{aligned} -\sqrt{80} \le -\sqrt{80-39y^2} &\le 0 \\ -10\sqrt{\frac{80}{39}} \cdot \sqrt{80} \le -10y\sqrt{80-39y^2} &\le 0 \\ 80 \le 50y^2 - 10y\sqrt{80-39y^2} + 80 &\le \frac{800}{\sqrt{39}} + \frac{400}{39} + 80 \\ \frac{80}{64} \le \frac{50y^2 - 10y\sqrt{80-39y^2} + 80}{64} &\le \frac{\frac{800}{\sqrt{39}} + \frac{400}{39} + 80}{64} \end{aligned}$$

Since $\frac{80}{64} \le s \le \frac{\frac{800}{\sqrt{39}} + \frac{400}{39} + 80}{64}$, the value could be computed.

$$\begin{aligned} \frac{1}{s_{\max}} + \frac{1}{s_{\min}} &= \frac{1}{\frac{\frac{800}{\sqrt{39}} + \frac{400}{39} + 80}{64}} + \frac{1}{\frac{80}{64}} \\ &= \boxed{\frac{64}{\frac{800}{\sqrt{39}} + \frac{400}{39} + 80} + \frac{64}{80}} \end{aligned}$$

□

Problem 5.

Find the greatest value of

$$x^2 + y^2$$

if x and y are real numbers satisfying

$$3x^2 + 2y^2 = 2x.$$

Solution y could be written in terms of x.

$$y^2 = \frac{2x - 3x^2}{2}$$

Because y is a real number, $2x - 3x^2 \geq 0$, or $0 \leq x \leq \frac{2}{3}$. Thus, the range could be utilized.

$$\begin{aligned} x^2 + y^2 &= x^2 + \frac{2x - 3x^2}{2} \\ &= \frac{-x^2 + 2x}{2} \end{aligned}$$

$$\begin{aligned} 0 \leq x &\leq \frac{2}{3} \\ -\frac{4}{9} \leq x^2 &\leq 0 \\ -\frac{2}{9} \leq \frac{x^2 + 2x}{2} &\leq \frac{1}{3} \end{aligned}$$

Thus, the maximum value is $\boxed{\frac{1}{3}}$. □

Problem 6.

Let x be any real number. Find the smallest value of

$$\frac{3x^2 + 6x + 5}{\frac{1}{2}x^2 + x + 1}.$$

Solution

$$\begin{aligned} \frac{3x^2 + 6x + 5}{\frac{1}{2}x^2 + x + 1} &= \frac{3(x+1)^2 + 2}{\frac{1}{2}(x+1)^2 + \frac{1}{2}} \\ &= \frac{6(x+1)^2 + 4}{(x+1)^2 + 1} \\ &= 6 - \frac{2}{(x+1)^2 + 1} \end{aligned}$$

Therefore, the minimum value $\boxed{4}$ is achieved when $x = -1$. □

Problem 7.

For $a, b \in \mathbb{R}$, find $a^2 + b^2$ if

$$a\sqrt{1 - b^2} + b\sqrt{1 - a^2} = 1.$$

Solution One possible solution is when $a = 0$ and $b = 1$. Thus, $a^2 + b^2 = \boxed{1}$. □