Problem 1.

Let $P(x)$ be a nonzero polynomial such that $(x-1)P(x+1) = (x+2)P(x)$ for every real x, and $P(2)^2 = P(3)$. Then $P\left(\frac{7}{2}\right) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution Substitute $x = -1, 0, 1$. Substituting the values leads to the fact that $P(-1) = P(0) = P(1) = 0$. Therefore, let

$$P(x) = f(x)(x+1)(x)(x-1).$$

Substituting to the equation and assuming that $x \neq -1, 0, 1$ leads to the following equations.

$$(x-1)f(x+1)(x+2)(x+1)(x) = (x+2)f(x)(x+1)(x)(x-1)$$
$$f(x+1) = f(x)$$

In other words, $f(x)$ is a constant for $x \neq -1, 0, 1$.

$$(f(2) \cdot 3 \cdot 2 \cdot 1)^2 = f(3) \cdot 4 \cdot 3 \cdot 2$$
$$f(2)^2 \cdot 36 = f(2) \cdot 24$$
$$f(2) = \frac{2}{3} \quad (\because f(2) \neq 0)$$

Substituting $x = \frac{7}{2}$ to $P(x)$, $P\left(\frac{7}{2}\right) = f\left(\frac{7}{2}\right)\left(\frac{7}{2}+1\right)\left(\frac{7}{2}\right)\left(\frac{7}{2}-1\right) = f(2) \cdot \frac{315}{8}$, which is $\frac{105}{4}$. Thus, $105 + 4 = \boxed{109}$. □

Problem 2.

For certain real number a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c$$

What is $f(1)$?

Solution $f(x)$ could be rewritten as $(x-m)(g(x))$. The coefficients of x^3, x^2, x, and x^0 could be compared.

$$\begin{aligned} f(x) &= (x-m)(x^3 + ax^2 + x + 10) \\ &= x^4 + (a-m)x^3 + (1-am)x^2 + (10-m)x - 10m \end{aligned}$$

Therefore, $m = -90$, $a = -89$, $b = -8009$, and $c = 900$. Thus, $f(1) = 1 + 1 + b + 100 + c = \boxed{-7007}$. □

Problem 3.

Let S be the number of ordered pairs of integers (a, b) with $1 \leq a \leq 100$ and $b \geq 0$ such that the polynomial $x^2 + ax + b$ can be factored into the product of two (not necessarily distinct) linear factors with integer coefficients. Find the remainder when S is divided by 1000.

Solution Let $-p$ and $-q$ be the roots of the polynomial $x^2 + ax + b$. In other words, $p + q = a$ and $pq = b$. Since the equations are symmetric, WLOG, let $p \geq q$.

p	q	Number of Cases
0	$1 \sim 100$	100
1	$1 \sim 99$	99
2	$2 \sim 98$	97
3	$3 \sim 97$	95
$\vdots$	$\vdots$	$\vdots$
50	50	1

Because an equation with two roots p and q is unique, all cases are counted. Thus, the total number of cases are $1+3+\cdots+97+99+100=\frac{100\cdot 50}{2}+100=2600$. Therefore, the answer is $\boxed{600}$.

□

Problem 4.

The polynomial $P(x)$ is cubic. What is the largest value of k for which the polynomials

$$Q_1(x)=x^2+(k-29)x-k \quad \text{and} \quad Q_2(x)=2x^2+(2k-43)x+k$$

are both factors of $P(x)$?

Solution Notice that because k is a constant, $P(x)$ could be rewritten.

$$\begin{aligned} P(x) &= Q_1(2ax+b)=(x^2+(k-29)x-k)(2ax+b) \\ &= Q_2(ax-b)=(2x^2+(2k-43)x+k)(ax-b) \end{aligned}$$

Therefore, the coefficients of x^2 and x could be compared for the value of k.

$$\begin{aligned} 2ak-58a+b &= 2ak-43a-2b \\ b &= 5a \end{aligned}$$

$$\begin{aligned} kb-29b-2ak &= -2bk+43b+ak \\ 3bk &= 3ak+72b \\ 15ak &= 3ak+360a \\ \therefore k &= \boxed{30} \end{aligned}$$

□

Problem 5.

A real number a is chosen randomly and uniformly from the interval $[-20,18]$. Calculate the probability that the roots of the polynomial

$$x^4+2ax^3+(2a-2)x^2+(-4a+3)x-2$$

are all real.

Solution Notice that when $x=1$ and $x=-2$, the polynomial is zero. Thus, with synthetic division, the polynomial could be factored.

$$(x-1)(x+2)(x^2+(2a-1)x+1)=0$$

For all the roots of x to be real, $(2a-1)^2-4$ must be greater than or equal to zero.

$$\begin{aligned} 4a^2-4a-3 &\geq 0 \\ (2a+1)(2a-3) &\geq 0 \\ -20\leq a\leq -\frac{1}{2}, 18\geq a &\geq \frac{3}{2} \end{aligned}$$

Lengths could be used to compute the probability.

$$\begin{aligned} \frac{\left(18-\frac{3}{2}\right)+\left(20-\frac{1}{2}\right)}{18+20} &= \frac{\frac{33}{2}+\frac{39}{2}}{38} \\ &= \frac{36}{38} \\ &= \boxed{\frac{18}{19}} \end{aligned}$$

□

Problem 6.

For what real values of k do

$$1988x^2 + kx + 8891 \quad \text{and} \quad 8891x^2 + kx + 1988$$

have a common zero?

Solution Let α be the common root.

$$1988\alpha^2 + k\alpha + 8891 = 0$$
$$8891\alpha^2 + k\alpha + 1988 = 0$$

Subtracting two equations, $6903\alpha^2 = 6903$ is obtained. In other words, the common root could be either 1 or -1. Therefore, k could be $\boxed{\pm 10879}$. $\square$