7.15.2025 AMC 10 Free Class Preview Homework Solution

Enoch Yu

July 2025

Problem

If the roots of the equation $2kx^2 - (3k+2)x + 2k - 1 = 0$ differ by 1, find the positive value of k.

Solution Let α and β be the roots of the equation. Therefore, $\alpha + \beta = \frac{3k-2}{2k}$ and $\alpha\beta = \frac{2k-1}{2k}$. Moreover, the following equations are true.

$$\begin{aligned}
\left(\frac{3k-2}{2k}\right)^2 - 4\left(\frac{2k-1}{2k}\right) &= 1 \\
\frac{9k^2-12k+4}{4k^2} - \frac{4k-2}{k} &= 1 \\
9k^2 - 12k + 4 - 16k^2 + 8k &= 4k^2 \\
11k^2 - 20k - 4 &= 0 \\
(11k+2)(k-2) &= 0 \\
\therefore k &= \boxed{2}
\end{aligned}$$

□

Problem

Solve the system of equations.

$$\begin{cases} x + xy + y &= 11 \\ x^2y + xy^2 &= 30 \end{cases}$$

Solution Let $a = x + y$ and $b = xy$. Therefore, $a + b = 11$ and $ab = 30$. Let a and b be the roots of the following equation.

$$t^2 - 11t + 30 = 0$$

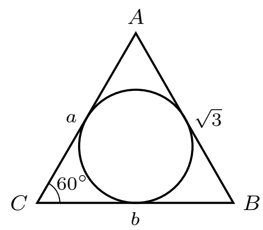
Therefore, $a = 5$ and $b = 6$, or $a = 6$ and $b = 5$. From the first case, the pairs of (x, y) are $(2, 3)$ and $(3, 2)$. From the second case, the pairs of (x, y) are $(1, 5)$ and $(5, 1)$. Therefore, the roots of the system of equations are $\boxed{(2,3),(3,2),(1,5),(5,1)}$. □

Hard Problems

Problem

A Circle O of radius r is inscribed to $\triangle ABC$ at D, E, and F. $\angle C = 60°$. The length of the side opposite to $\angle C$ is $c = \sqrt{3}$. Find the range of r.

Solution



From the diagram, notice that the values of $a+b$ and ab could be written in terms of r.

$$\begin{aligned} a - \tan 60^\circ \cdot r + b - \tan 60^\circ \cdot r &= \sqrt{3} \\ a + b &= \sqrt{3}(2r+1) \\ \frac{r(a+b+\sqrt{3})}{2} &= \frac{\sqrt{3}ab}{4} \\ \frac{r\left(\sqrt{3}(2r+1)+\sqrt{3}\right)}{2} &= \frac{\sqrt{3}ab}{4} \\ ab &= 2r(2r+2) \end{aligned}$$

Let a and b be the root of the equation

$$t^2 - \sqrt{3}(2r+1) + 4r^2 + 4r = 0$$

Discriminant could be used since a and b are positive numbers.

$$\begin{aligned} 12r^2 + 12r + 3 - 16r^2 - 16r &\geq 0 \\ 4r^2 + 4r - 3 &\leq 0 \\ (2r+3)(2r-1) &\leq 0 \\ \therefore \boxed{0 < r \leq \frac{1}{2}} \end{aligned}$$

□

Problem

Show that there is one and only one of real numbers x, y, and z that is at least $\sqrt[3]{4}$ if $x+y+z=0$ and $xyz=1$.

Proof. WLOG, let $x \leq y < z$ and let x and y be the solution to the following equation.

$$t^2 + zt + \frac{1}{z} = 0$$

Using discriminant, $z^2 - \frac{4}{z} \geq 0$. Because z must be a positive number, $z^3 \geq 4$. In other words, only $z \geq \sqrt[3]{4}$. □

Problem

Solve for real x, y, and z.

$$\begin{cases} x+y &= 2 \\ xy - z^2 &= 1 \end{cases}$$

Solution Notice let x and y be the solution to the following equation.

$$t^2 - 2t + 1 + z^2 = 0$$

Using discriminant, it is evident that $z=0$ and $z=y=1$. Thus, the solution is $\boxed{(1,1,0)}$. □