

# 7.15.2025 AMC 10,12 Free Class Preview Homework

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## Problem

Show that there is one and only one of real numbers  $x$ ,  $y$ , and  $z$  that is at least  $\sqrt[3]{4}$  if  $x + y + z = 0$  and  $xyz = 1$ .

*Proof.* WLOG, let  $x \leq y < z$  and let  $x$  and  $y$  be the solution to the following equation.

$$t^2 + zt + \frac{1}{z} = 0$$

Using discriminant,  $z^2 - \frac{4}{z} \geq 0$ . Because  $z$  must be a positive number,  $z^3 \geq 4$ . In other words, only  $z \geq \sqrt[3]{4}$ .  $\square$

## Problem

Solve for real  $x$ ,  $y$ , and  $z$ .

$$\begin{cases} x + y &= 2 \\ xy - z^2 &= 1 \end{cases}$$

**Solution** Notice let  $x$  and  $y$  be the solution to the following equation.

$$t^2 - 2t + 1 + z^2 = 0$$

Using discriminant, it is evident that  $z = 0$  and  $z = y = 1$ . Thus, the solution is  $\boxed{(1, 1, 0)}$ .  $\square$

## Problem

Find the greatest real value of  $z$  such that  $x + y + z = 5$  and  $xy + yz + zx = 3$ .  $x$  and  $y$  are real numbers.

**Solution** Vieta's formulas could be used to solve the problem.

$$\begin{aligned} x + y &= 5 - z \\ xy + z(x + y) &= 3 \\ xy &= 3 - z(5 - z) = z^2 - 5z + 3 \end{aligned}$$

WLOG, let  $x$  and  $y$  be the roots of the following equation.

$$t^2 + (z - 5)t + z^2 - 5z + 3 = 0$$

Using discriminant,

$$\begin{aligned} z^2 - 10z + 25 - 4z^2 + 20z - 12 &\geq 0 \\ 3z^2 - 10z - 13 &\leq 0 \\ (3z - 13)(z + 1) &\leq 0 \\ -1 &\leq z \leq \frac{13}{3} \end{aligned}$$

Since all three variables satisfies the equivalent inequality, the greatest real value of  $z$  is  $\boxed{\frac{13}{3}}$ . □

**Problem**

Show that  $1 < a + b < \frac{4}{3}$  and  $\frac{8}{9} < a^2 + b^2 < 1$  if  $a + b + c = a^2 + b^2 + c^2 = 1$ , and  $a > b > c$ .

*Proof.* Let  $x = a - c$  and  $y = b - c$ . Therefore,  $x > y$ . Moreover, the following equations are true.

$$\begin{aligned} x + y &= a + b - 2c \\ &= 1 - 3c \\ xy &= (a - c)(b - c) \\ &= ab - c(a + b) + c^2 \\ &= -2c(a + b) + c^2 \\ &= -2c(1 - c) + c^2 \end{aligned}$$

Let  $x$  and  $y$  be the solution for the following equation.

$$t^2 - (1 - 3c)t - 2c(1 - c) + c^2 = 0$$

Therefore, the following conditions must satisfy.

$$\begin{cases} 1 - 3c > 0 \\ -2c(1 - c) + c^2 > 0 \\ (1 - 3c)^2 + 8c(1 - c) - 4c^2 > 0 \end{cases}$$

Therefore,  $c < \frac{1}{3}$ ,  $c < 0$  or  $c > \frac{2}{3}$ , and  $-\frac{1}{3} < c < 1$ . In other words,  $-\frac{1}{3} < c < 0$ .

$$\begin{aligned} -\frac{1}{3} < 1 - a - b < 0 & & 0 < c^2 < \frac{1}{9} \\ \therefore 1 < a + b < \frac{4}{3} & & 0 < 1 - a^2 - b^2 < \frac{1}{9} \\ & & \therefore \frac{8}{9} < a^2 + b^2 < 1 \end{aligned}$$

□

**Problem**

Find the range of the function  $y = \sqrt{2 - x} + \sqrt{x - 1}$ .

**Solution I** Let  $a = \sqrt{2 - x}$  and  $b = \sqrt{x - 1}$ . Therefore,  $a + b = y$  and  $(a + b)^2 = 1 + 2\sqrt{(2 - x)(x - 1)} = y^2$ . In other words,  $ab = \frac{y^2 - 1}{2}$ .

Let  $a$  and  $b$  be the roots of the equation  $t^2 - yt + \frac{y^2 - 1}{2} = 0$ . Using discriminant,  $y^2 - 2y^2 + 2 \geq 0$ , or  $-\sqrt{2} \leq y \leq \sqrt{2}$ . However, notice that  $y > 0$  and  $y \leq -1$  or  $y \geq 1$ . Therefore, the range for  $y$  is  $\boxed{[-1, 1]}$ . □

**Solution II** First, notice that  $1 \leq x \leq 2$ . Moreover, the function is a polynomial. Therefore, the following equations are true.

$$\begin{aligned} y' &= \frac{-1}{\sqrt{2 - x}} + \frac{1}{\sqrt{x - 1}} \\ y'' &= \frac{-1}{2\sqrt{(2 - x)^3}} - \frac{1}{\sqrt{(x - 1)^3}} < 0 \end{aligned}$$

In other words,  $y$  is a concave down function that has a critical point at  $x = 1.5$ . Therefore, the minimum value of  $y$  is 1 and the maximum is  $\sqrt{2}$ . Thus,  $\boxed{1 \leq y \leq \sqrt{2}}$ . □

## Problem

How many distinct ordered triples  $(x, y, z)$  satisfy the following equations?

$$\begin{aligned}x + 2y + 4z &= 12 \\xy + 4yz + 2xz &= 22 \\xyz &= 6\end{aligned}$$

**Solution** First and foremost, notice that the problem might have been more affable if the coefficients of the variables in the second equation were different. If a pattern is found, any equation could be forcibly changed for the pattern to satisfy flawlessly.

$$\begin{aligned}x + 2y + 4z &= 12 \\2xy + 8yz + 4xz &= 44 \\8xyz &= 48\end{aligned}$$

Let  $a = x$ ,  $b = 2y$ , and  $c = 4z$ . Then, the following equations are true.

$$\begin{aligned}a + b + c &= 12 \\ab + bc + ca &= 44 \\abc &= 48\end{aligned}$$

Using, Vieta's formula, it is evident that  $a$ ,  $b$ , and  $c$  are the real roots of the following equation.

$$t^3 - 12t^2 + 44t - 48 = 0$$

Solving for  $t$  will lead to the values of  $a$ ,  $b$ , and  $c$ .

$$\begin{aligned}t^3 - 12t^2 + 44t - 48 &= 0 \\(t - 2)(t - 4)(t - 6) &= 0 \\t &= 2, 4, 6\end{aligned}$$

Because 2, 4, 6 are all different numbers, there exists  $3!$ , or  $\boxed{6}$ , distinct ordered triples  $(x, y, z)$ . □