7.11.2025 AMC 10, 12 Free Class

Enoch Yu

11 July 2025

Problem 1

Find all pairs of integers (x, y) such that $x^3 + y^3 = (x + y)^2$.

Solution

$$(x + y)(x^2 - xy + y^2) = (x + y)^2$$

$$\begin{cases} x + y = 0 \implies x = -y \\ x + y \neq 0 \implies x^2 - xy + y^2 = x + y \end{cases}$$

Since the first case provided the solution, the second case could be observed.

$$\begin{aligned} x^2 - xy + y^2 &= x + y \\ x^2 - x(y + 1) + y^2 - y &= 0 \\ x &= \frac{y + 1 \pm \sqrt{y^2 + 2y + 1 - 4y^2 + 4y}}{2} \\ D &= -3y^2 + 6y + 1 \geq 0 \\ &\Rightarrow 3y^2 - 6y - 1 \leq 0 \\ -1 < \frac{3 - 2\sqrt{3}}{3} &\leq y \leq \frac{3 + 2\sqrt{3}}{3} < 3 \\ \therefore y &= 0, 1, 2 \Rightarrow (1, 0), (2, 1), (1, 2), (2, 2) \end{aligned}$$

Thus, $\boxed{(x, -x), (1, 0), (2, 1), (1, 2), (2, 2)}$ are the roots. □

Problem 2

Solve the equation $x^3 - 3x = \sqrt{x + 2}$ for all real values of x.

Solution Notice that when $x = 2$, the equation satisfies. Therefore, intuitively, the cases for x could be set.

$$\begin{cases} x \leq 2 \implies \sqrt{x + 2} \geq x \\ x > 2 \implies \sqrt{x + 2} < x \end{cases}$$

Observe the second equation.

$$\begin{aligned} x^3 - 3x - \sqrt{x + 2} &= 0 \\ x^3 - 3x - x &< 0 \\ x(x^2 - 4) &< 0 \\ \therefore x < -2 \text{ or } 0 &< x < 2 \end{aligned}$$

Due to a contradiction, the fact that the roots of the equation are never greater than 2 could be deduced. Moreover, since x is a real number, $-2 \leq x \leq 2$. Let $x = 2\cos\theta$. Therefore, the following equations could be deduced.

$$\begin{aligned} 8\cos^3\theta - 6\theta &= \sqrt{2\cos\theta + 2} \\ \cos 3\theta &= \sqrt{\frac{\cos\theta + 1}{2}} \\ \cos 3\theta &= \cos\frac{\theta}{2} \\ \therefore x &= \boxed{2, 2\cos\frac{4\pi}{5}, 2\cos\frac{4\pi}{7}} \end{aligned}$$

□