7.8.2025 AMC 10, 12 Free Class Preview Homework Solution

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July 2025

Problem 1.

Solve $\sqrt{2x+1} + 2\sqrt{x-1} = 4$.

Solution Let $a = \sqrt{2x+1}$ and $b = \sqrt{x-1}$ where $b \le 2$. Therefore, $a^2 - 2b^2 = 3$ and $a + 2b = 4$. Substitution could be utilized.

$$\begin{aligned} a &= 4 - 2b \\ (4-2b)^2 - 2b^2 &= 3 \\ 2b^2 - 16b + 13 &= 0 \\ \therefore b &= \frac{8-\sqrt{38}}{2} \end{aligned}$$

In other words, $\sqrt{x-1} = \frac{8-\sqrt{38}}{2}$. Meaning, $x = \frac{102-16\sqrt{38}}{4} + 1$, or $\frac{53-8\sqrt{38}}{2}$.

Lemma. *There exists at most one real root to the equation.*

Proof. Let $f(x) = \sqrt{2x+1} + 2\sqrt{x-1} - 4$.

$$\begin{aligned} f'(x) &= \frac{2}{2\sqrt{2x+1}} + \frac{2}{2\sqrt{x-1}} \\ &= \frac{1}{\sqrt{2x+1}} + \frac{1}{\sqrt{x-1}} \end{aligned}$$

Since $f'(x)$ is always positive wherever it is defined, function $f(x)$ is always increasing. In other words, there could be at most one real root to the equation. □

Therefore, $x = \frac{53-8\sqrt{38}}{2}$ is the only real roots. □

Problem 2.

Solve $\sqrt{x - \frac{5}{x}} - \sqrt{5 - \frac{5}{x}} = x$.

Solution

Key Word Average Method

Notice that every number could be written as a sum of two numbers that have same distance from the half the value of the number. Therefore, let $\sqrt{x - \frac{5}{x}} = \frac{x}{2} + k$ and $-\sqrt{5 - \frac{5}{x}} = \frac{x}{2} - k$.

$$\begin{aligned}
\sqrt{x - \frac{5}{x}} &= \frac{x}{2} + k \\
-\sqrt{5 - \frac{5}{x}} &= \frac{x}{2} - k \\
x - \frac{5}{x} &= \frac{x^2}{4} + kx + k^2 \\
5 - \frac{5}{x} &= \frac{x^2}{4} - kx + k^2 \\
x - 5 &= 2kx \\
\therefore k &= \frac{x-5}{2x}
\end{aligned}$$

Substitution could be used.

$$\begin{aligned}
\sqrt{x - \frac{5}{x}} &= \frac{x}{2} + \frac{x-5}{2x} \\
2\sqrt{x - \frac{5}{x}} &= x + \frac{x-5}{x} \\
2\sqrt{x - \frac{5}{x}} &= 1 + \left(x - \frac{5}{x}\right) \\
1 + -2\sqrt{x - \frac{5}{x}} + \left(x - \frac{5}{x}\right) &= 0 \\
(1 - \sqrt{x - \frac{5}{x}})^2 &= 0 \\
x - \frac{5}{x} &= 1 \\
\therefore x &= \boxed{\frac{1-\sqrt{21}}{2}}
\end{aligned}$$

Through graphing, it is evident that the x value is the only possible x. □

Problem 3.

Find $x^6 - 2\sqrt{2}x^5 - 3x^4 - x^3 + 2\sqrt{5}x^2 - 4x + \sqrt{5}$ if $x = \sqrt{5} + \sqrt{2}$.

Solution

$$\begin{aligned}
& x^6 - 2\sqrt{2}x^5 - 3x^4 - x^3 + 2\sqrt{5}x^2 - 4x + \sqrt{5} \\
&= x^5\left(x - 2\sqrt{2}\right) - 3x^4 + x^2\left(-x + 2\sqrt{5}\right) - 4x + \sqrt{5} \\
&= \left(\sqrt{5} - \sqrt{2}\right)(x^5 + x^2) - 3x^4 - 4x + \sqrt{5} \\
&= x\left(\sqrt{5} - \sqrt{2}\right)(x^4 + x) - 3(x^4 + x) - x + \sqrt{5} \\
&= \left(\sqrt{5} + \sqrt{2}\right)\left(\sqrt{5} - \sqrt{2}\right) - 3(x^4 + x) - \sqrt{2} \\
&= \boxed{\sqrt{2}}
\end{aligned}$$

□

Problem 4.

Show that $\sqrt[3]{ax^2+by^2+cz^2} = \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}$ if $ax^3 = by^3 = cz^3$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$.

Proof.

$$\begin{aligned}
\sqrt[3]{ax^2+by^2+cz^2} &= \sqrt[3]{\frac{ax^3}{x}+\frac{by^3}{y}+\frac{cz^3}{z}} \\
&= \sqrt[3]{ax^3} \\
&= \sqrt[3]{ax^3}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \\
&= \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}
\end{aligned}$$

□