7.1.2025 AMC 10 Free Class Preview Homework Solution

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Problem

Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 3, the value of the fraction is increased by 30%?

Solution First and foremost, the equation could be written to manifest the situation.

$$\frac{x+3}{y+3} = \frac{x}{y} \cdot \frac{130}{100}$$

The equation could be simplified through rearrangement.

$$\begin{aligned} \frac{x+3}{y+3} &= \frac{x}{y} \cdot \frac{130}{100} \\ 10xy + 30y &= 13xy + 39x \\ 3xy + 39x - 30y &= 0 \\ xy + 13x - 10y &= 0 \\ (x-10)(y+13) &= -130 \end{aligned}$$

Notice that x and y are relatively prime positive integers.

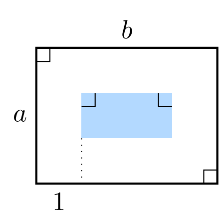
$x-10$	$y+13$	x	y	Validity
-1	130	9	143	Y
-2	65	8	78	N
-5	26	5	39	Y

Therefore, $\boxed{\text{two}}$ fractions $\frac{9}{143}$ and $\frac{5}{39}$ satisfy the conditions. □

Problem

A rectangular floor measures a by b feet, where a and b are positive integers with $b > a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the sides of the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half of the area of the entire floor. How many possibilities are there for the ordered pair (a, b)?

Solution First and foremost, the diagram could be drawn.



Writing the equation,

$$\begin{aligned}(a-2)(b-2) &= \frac{1}{2}ab \\ 2ab - 4a - 4b + 8 &= ab \\ (a-4)(b-4) &= 8\end{aligned}$$

Notice that a and b are positive integers. Therefore, there exists $\boxed{\text{two}}$ order pairs $(3, 10)$ and $(4, 6)$ that satisfy the condition. □

Challenge Problems

Problem

The integer N is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}$$

Prove that N is a perfect square.

Proof. First and foremost, rearrange the terms.

$$\begin{aligned}\frac{1}{x} + \frac{1}{y} &= \frac{1}{N} \\ \frac{x+y}{xy} &= \frac{1}{N} \\ N(x+y) &= xy \\ (x-N)(y-N) &= N^2\end{aligned}$$

Notice that because $x - N$ and $y - N$ are reflective, the number of ordered pairs (x, y) must be even, unless there is a double counting. Double counting only occurs when $x - N = y - N$. In other words, there must be a factor p where $p^2 = N$ for there to be an odd number of ordered pairs. □

Problem

m, n are integers such that

$$m^2 + 3m^2n^2 = 30n^2 + 517.$$

Find $3m^2n^2$.

Solution Simon's favorite factorization could be utilized.

$$\begin{aligned}3m^2n^2 + m^2 - 30n^2 - 517 &= 0 \\ (m^2 - 10)(3n^2 + 1) &= 507 = 3 \cdot 13^2\end{aligned}$$

Because m and n are integers, $m^2 - 10$ must be 39 while $3n^2 + 1$ is 13. Therefore, $49 \cdot 12 = \boxed{588}$. □

Problem

There are two consecutive whole numbers such that the difference of their cubes is equal to 181 squared. Find the sum of these two numbers.

Solution Let the two consecutive whole numbers be a and $a+1$.

$$\begin{aligned}(a+1)^3 - a^3 &= 181^2\\ 3a^2+3a+1 &= 181^2\\ 3a^2+3a &= 182\cdot 180\\ a^2+a-182\cdot 60 &= 0\\ a^2+a-10920 &= 0\\ 100 < \sqrt{10920} &< 105\\ 10920 &= 104\cdot 105\\ (a+105)(a-104) &= 0\end{aligned}$$

Because a is a whole number, $a=104$. Therefore, $104+105 = \boxed{209}$. □

Problem

There exist unique positive integers x and y that satisfy the equation $x^2+84x+2008=y^2$. Find $x+y$.

Solution

$$\begin{aligned}x^2+84x+1764+244 &= y^2\\ (x+42)^2+244 &= y^2\\ (x+y+42)(x-y+42) &= -244\end{aligned}$$

Notice that y must be greater than $x+42$. Moreover, $x+y+42=122$ and $x-y+42=-2$ is a reasonable approach. Other cases does not work because x and y are not natural numbers. Therefore, $y=62$ and $x=18$. Thus, $x+y=\boxed{80}$. □