§ Solutions for Farmer's Problems (Activity Available Here)

Problem 1

What is the number of possible words that can form with the letters H, A, V, E, N?

(A) 60 (B) 120 (C) 25 (D) 5

Solution: Because the letters are all distinct and no replacement is allowed, the number of possible words from the five letters can be represented as $5! = 120$. □

Problem 2

Possible words with B, A, N, A, N, A??

(A) 36 (B) 120 (C) 720 (D) 60

Solution: Unlike the previous problem, not all letters are distinct. Therefore, the arrangements of the same letters must be divided from 6!. Therefore, the total of $\frac{6!}{3!2!} = 60$ words can form with the letters. □

Problem 3 (2015 AMC 8 Problem 10)

The notorious farmer needs some help with counting! How many integers between 1000 and 9999 have four distinct digits?

(A) 4536 (B) 3024 (C) 5040 (D) 6480

Solution: Notice that there exists 9 possible digits for the thousands place, from 1 to 9. Because the digits are distinct, there exists $10 - 1 = 9$ possible numbers for the hundreds place. Similarly, 8 and 7 digits are possible are possible for the tens and ones place respectively. Therefore, there exists $9 \cdot 9 \cdot 8 \cdot 7 = 4536$ four digit integers with four distinct digits. □

Problem 4 (2017 AMC 8 Problem 20)

This time, the farmer wants to know the probability that a randomly chosen integer between 1000 and 9999 is an odd integer whose digits are all distinct!

(A) $\frac{107}{400}$ (B) $\frac{14}{75}$ (C) $\frac{56}{225}$ (D) $\frac{7}{25}$

Solution: Note that only five digits $(1, 3, 5, 7, 9)$ are possible for the ones place. Similarly, there exists $9 - 1 = 8$ possible digits for the thousands place. Continuing, $10 - 2 = 8$ and 7 numbers are possible for the hundreds and ones digits respectively.

Therefore, the probability of choosing an odd integer between 1000 and 9999 with four distinct digits is $\frac{8 \cdot 8 \cdot 7 \cdot 5}{9 \cdot 10 \cdot 10 \cdot 10} = \frac{56}{225}$. □

Problem 5 (1985 AJHSME Problem 15)

The farm animals love the number 2. Therefore, the farmer decided to count the number of integers between 100 and 400 that contains the number 2. Can you help him with the counting?

(A) 100 (B) 138 (C) 140 (D) 148

Solution: The cases can be divided by the hundreds digit. From 100 to 199, there exists $10 + 9 = 19$ numbers with the number 2. From 200 to 299, there exists 100 numbers. Continuing from 300 to 399, there exists 19 numbers. Therefore, the total number of integers that contains the number 2 from 100 to 300 is $19 + 100 + 19 = 138$. □

Problem 6

Swanky the donkey had a beef with the farmer due to low compensation for work. Swanky, being a swanky donkey, offered a mathematical game to the farmer to demand an increase in "celery". The rule is simple. The farmer is given a fair coin where he flips the coin in every turn. If the coin lands on tails, the farmer moves one step to the left. Where as if the coin lands on heads, the farmer moves one step to the right.

The farm has a fence at position $x = 0$ and a garden of nuggets at position $x = 10$. If the farmer ever hits the fence, he loses, and Swanky receives the raise. On the other hand, Swanky loses if the farmer reaches the nugget garden.

What is the probability that the farmer loses if he starts from position $x = 4$?

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{9}{25}$

Solution: Let $f(x)$ represent the probability that the farmer hits the fence from the position x. By definition, $f(0) = 1$ and $f(10) = 1$ satisfy. Moreover, because the coin is a fair coin, the following equation can be obtained.

$$f(x) = \frac{1}{2}f(x-1) + \frac{1}{2}f(x+1)$$

Rearranging the equation, the following recursive equation for $f(x)$ satisfy.

$$f(x+1) = 2f(x) - f(x-1)$$

Different values can be substituted to notice the pattern of the formula.

$$\begin{aligned} f(2) &= 2f(1) - f(0) = 2f(1) \\ f(3) &= 2f(2) - f(1) = 3f(1) \\ f(4) &= 2f(3) - f(2) = 4f(1) \end{aligned}$$

Consider the following lemma.

Lemma 1.1. *$f(x) = xf(1)$ for all $x > 1$.*

Proof. Substitute $f(x) = xf(1)$ to the original recursive equation.

$$\begin{aligned} (x+1)f(1) &= 2xf(1) - (x-1)f(1) \\ \therefore (x+1)f(1) &= (x+1)f(1) \end{aligned}$$

Because substituting the equation result in the same conclusion, $f(x) = xf(1)$ is a solution to the recursive formula. □

Because $f(10) = 1$ is true by definition, $f(10) = 10f(1) = 1$. Therefore, $f(1) = \frac{1}{10}$. Continuing, $f(4) = 4 \cdot \frac{1}{10} = \frac{2}{5}$. In other words, the probability that the farmer loses is $\frac{2}{5}$. □

Problem 7

Hmmm... Swanky is still not satisfied with the probability. This time, Swanky decided to give the farmer an unfair coin with $p = 0.6$ as the probability of landing on tails. Swanky being swanky, he decided to move the garden of nuggets to $x = 1000$.

If the farmer start from the position $x = 600$, what is the probability that Swanky receives the raise?

(A) 0.6 **(B)** $1 - \epsilon$ **(C)** 0.36 **(D)** 1

Solution: Let $f(x)$ represent the probability that Swanky wins from position x. By definition, $f(0) = 0$ and the following equation satisfy.

$$f(x) = pf(x+1) + (1-p)\,f(x-1)$$

Rearranging and shifting the relation, the following equation can be obtained.

$$f(x) = \frac{f(x-1) - f(x-2)}{p} + f(x-2)$$

The equation can further be simplified by subtracting $f(x-1)$ from both sides.

$$f(x) - f(x-1) = \frac{f(x-1) - f(x-2)}{p} + f(x-2) - f(x-1)$$

Continuing, let $g(x) = f(x) - f(x-1)$ for $x > 1$. Therefore,

$$g(x) = g(x-1)\left(\frac{1}{p} - 1\right) = \frac{g(x-1)}{p} - g(x-1) = g(x-1)\alpha$$

is true where $\alpha = \frac{1}{p} - 1 = \frac{2}{3}$. Substituting different values for $g(x)$, the following equations are obtained.

$$\begin{aligned} g(2) &= f(1)\alpha \\ g(3) &= f(2)\alpha = f(1)\alpha^2 \\ g(4) &= f(3)\alpha = f(1)\alpha^3 \end{aligned}$$

Therefore, one could notice that $g(x) = f(1)\alpha^{x-1}$. Moreover, by definition of $g(x)$, the following equation satisfy.

$$f(x) = f(x-1) + g(x) = f(x-1) + f(1)\alpha^{x-1}$$

Similarly, substituting different values for $f(x)$ provides the following patterns.

$$\begin{aligned} f(2) &= f(1) + f(1)\alpha \\ f(3) &= f(2) + f(1)\alpha^2 = f(1) + f(1)\alpha + f(1)\alpha^2 \\ f(4) &= f(3) + f(1)\alpha^3 = f(1) + f(1)\alpha + f(1)\alpha^2 + f(1)\alpha^2 \end{aligned}$$

Therefore, it is evident that the following equation is true for $f(x)$.

$$f(x) = f(1)\left(1 + \alpha + \alpha^2 + \cdots + \alpha^{x-1}\right) = f(1) \cdot \frac{1 - \alpha^x}{1 - \alpha}$$

To further simplify the equation, the value of $f(1)$ could be represented in terms of α and x.

Note that because $f(1000) = 1$, $f(1000) = f(1) \cdot \frac{1-\alpha^{1000}}{1-\alpha} = 1$ and $f(1) = \frac{1-\alpha}{1-\alpha^{1000}}$. Thus, the following equation is true.

$$f(x) = \frac{1-\alpha}{1-\alpha^{1000}} \cdot \frac{1-\alpha^x}{1-\alpha} = \frac{1-\alpha^x}{1-\alpha^{1000}} = \frac{\alpha^x - 1}{\alpha^{1000} - 1}$$

Therefore, $f(600) = \frac{\alpha^{600}-1}{\alpha^{1000}-1}$ is true where $\alpha = \frac{2}{3}$. Substituting the value, $f(600) \approx 1$. In other words, Swanky will almost always win the game. □

Problem 8 (1974 AHSME Problem 24)

After losing the game with Swanky, the notorious farmer decided to study probability. Can you help him find the probability of rolling a fair die six times and getting at least a five at least five times?

(A) $\frac{2}{729}$ (B) $\frac{3}{729}$ (C) $\frac{12}{729}$ (D) $\frac{13}{729}$

Solution: Notice that the probability of rolling at least a five is $\frac{2}{6} = \frac{1}{3}$. Therefore, the probability of rolling at least a five six times is $\left(\frac{1}{3}\right)^6 = \frac{1}{729}$. Similarly, the probability of rolling at least a five five times is $\frac{2}{3} \cdot \left(\frac{1}{3}\right)^6 \cdot 6 = \frac{12}{729}$. Therefore, the probability of rolling at least a five at least five times is $\frac{1}{729} + \frac{12}{729} = \frac{13}{729}$. □