Trigonometric Identities

Enoch Yu

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24. $\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$
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26. $\sin^2\alpha=\frac{1}{2}(1-\cos 2\alpha)$
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2 Angle Sum Relationships

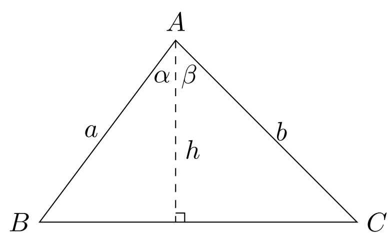
2.1 List of Relationships

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

2.2 Relationship for $\sin(\alpha+\beta)$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Proof. Consider the diagram below.



It is evident that $h = a\cos\alpha = b\cos\beta$ because $\cos\alpha = \frac{a}{y}$ and $\cos\beta = \frac{b}{y}$. Using the area formula of triangle, the following equation is true.

$$\frac{1}{2}ab\sin(\alpha+\beta) = \frac{1}{2}ah\sin\alpha + \frac{1}{2}bh\sin\beta$$

h could be replaced with $b\cos\beta$ and $a\cos\alpha$ respectively.

$$\frac{1}{2}ab\sin(\alpha+\beta) = \frac{1}{2}ab\cos\beta\sin\alpha + \frac{1}{2}ba\cos\alpha\sin\beta$$

By multiplying both sides by two and dividing by ab (because $ab \neq 0$), the following relationship may be obtained. Moreover, despite the fact that the current proof is not feasible for the case where $ab = 0$, the relationship appears to be true for $ab = 0$ also.

$$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

□

2.3 Relationship for $\cos(\alpha+\beta)$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Proof. Let $\alpha = \frac{\pi}{2} + \gamma$. α could be substituted for $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. The following equation is obtained.

$$\sin(\frac{\pi}{2}+\gamma+\beta) = \sin(\frac{\pi}{2}+\gamma)\cos\beta + \cos(\frac{\pi}{2}+\gamma)\sin\beta$$

Using the properties of trigonometric functions, $\sin(\frac{\pi}{2}+\gamma) = \cos\gamma$ and $\cos(\frac{\pi}{2}+\gamma) = -\sin\gamma$ are true.

$$\therefore \sin(\frac{\pi}{2}+\gamma)\cos\beta + \cos(\frac{\pi}{2}+\gamma)\sin\beta = \cos\gamma\cos\beta - \sin\gamma\sin\beta$$

In another words, $\sin(\frac{\pi}{2}+\gamma+\beta) = \cos(-\gamma-\beta) = \cos(\gamma+\beta) = \cos\gamma\cos\beta - \sin\gamma\sin\beta$ is true. The equation could be rewritten by changing the variables to adhere to convention in using variables:

$$\therefore \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

□

2.4 Relationship for $\tan(\alpha + \beta)$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Proof. $\tan(\alpha+\beta)$ could be rewritten as $\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$ by definition. Moreover, because $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, it is evident that

$$\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

is true. $\cos\alpha\cos\beta$ could be factored from both the numerator and the denominator.

$$\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \frac{(\cos\alpha\cos\beta)(\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta})}{(\cos\alpha\cos\beta)(1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta})}$$

The terms may be organized with different forms.

$$\frac{(\cos\alpha\cos\beta)(\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta})}{(\cos\alpha\cos\beta)(1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta})} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Therefore, the following expression is true.

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

□

3 Angle Difference Relationships

3.1 List of Relationships

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

3.2 Relationship for $\sin(\alpha - \beta)$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Proof. The equation $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ is proven to be true.

Substitute $-b$ instead of b to obtain $\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

$$\therefore \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

□

3.3 Relationship for $\cos(\alpha - \beta)$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Proof. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ is proven to be true.

Substitute $-b$ instead of b to obtain $\cos(\alpha-\beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

$$\therefore \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

□

3.4 Relationship for $\tan(\alpha - \beta)$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Proof. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ is proven to be true.

Substitute $-b$ instead of b to obtain $\tan(\alpha - \beta) = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)} = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$.

$$\therefore \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

□

4 Double Angle Relationships

4.1 List of Relationships

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha \\ &= \frac{2\tan\alpha}{1 + \tan^2\alpha}\end{aligned}$$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha \\ &= \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha}\end{aligned}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

4.2 Relationship for $\sin 2\alpha$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

Proof. Substitute α instead of β in the proven equation: $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

$$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$$

Therefore, the following equation is true.

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

□

$$\sin 2\alpha = \frac{2\tan\alpha}{1 + \tan^2\alpha}$$

Proof. $\tan 2\alpha$ could be rewritten to use the formulas that would later be proven.

$$\tan 2\alpha = \frac{\sin(2\alpha)}{\cos(2\alpha)} = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$\cos 2\alpha$ could be multiplied to both sides.

$$\sin 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}\cdot\cos 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}\cdot\frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

The following equation is resulted.

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$$

□

4.3 Relationship for $\cos 2\alpha$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\\ &= 2\cos^2\alpha - 1\\ &= 1 - 2\sin^2\alpha\end{aligned}$$

Proof. Substitute α instead of β in the proven equation: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$\cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$$

Therefore, the following equation is true.

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

Using Pythagorean Trigonometric Identity, the equation could be further modified.

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

□

$$\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

Proof. It is evident that $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$ is true. The equation could be modified to factor $\cos^2\alpha$ from both numerator and denominator.

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\\ &= \frac{\cos^2\alpha - \sin^2\alpha}{1}\\ &= \frac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha + \sin^2\alpha}\\ &= \frac{(\cos^2\alpha)(1-\frac{\sin^2\alpha}{\cos^2\alpha})}{(\cos^2\alpha)(1+\frac{\sin^2\alpha}{\cos^2\alpha})}\\ &= \frac{1-\tan^2\alpha}{1+\tan^2\alpha}\end{aligned}$$

$$\therefore \cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

□

4.4 Relationship for $\tan 2\alpha$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Proof. Substitute α instead of β in the proven equation: $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

$$\tan(\alpha+\alpha) = \frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}$$

Therefore, the following equation is true.

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

□

5 Triple Angle Formulas

5.1 List of Formulas

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$
$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

Addition and double angle formulas may be utilized to derive triple angle formulas.

5.2 Formula for $\sin 3\alpha$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

Proof. $\sin 3\alpha$ may be rewritten as $\sin(2\alpha+\alpha)$. $\sin(2\alpha+\alpha) = \sin(2\alpha)\cos\alpha + \cos(2\alpha)\sin\alpha$ could be obtained by using addition relationship. The expression could be further modified through using double angle relationship.

$$\sin(2\alpha)\cos\alpha + \cos(2\alpha)\sin\alpha = 2\sin\alpha\cos^2\alpha + (\cos^2\alpha - \sin^2\alpha)\sin\alpha$$

$2\sin\alpha\cos^2\alpha + (\cos^2\alpha - \sin^2\alpha)\sin\alpha = 3\sin\alpha\cos^2\alpha - \sin^3\alpha$ could be obtained through rearrangement of terms. Pythagorean trigonometric identities could be utilized to further simplify the equation by using only $\sin\alpha$.

$$\begin{aligned} 3\sin\alpha\cos^2\alpha - \sin^3\alpha &= 3\sin\alpha(1-\sin^2\alpha) - \sin^3\alpha \\ &= 3\sin\alpha - 4\sin^3\alpha \end{aligned}$$

Therefore, the following equation is true.

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

□

5.3 Formula for $\cos 3\alpha$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

Proof. $\cos\alpha$ may be rewritten as $\cos(2\alpha+\alpha)$. $\cos(2\alpha+\alpha) = \cos(2\alpha)\cos\alpha + \sin(2\alpha)\sin\alpha$ could be obtained by using addition relationship. The expression could be further modified through using double angle relationship.

$$\cos(2\alpha)\cos\alpha + \sin(2\alpha)\sin\alpha = (\cos^2\alpha - \sin^2\alpha)\cos\alpha - 2\sin^2\alpha\cos\alpha$$

Pythagorean trigonometric identities could be utilized to further simplify the equation by using only $\cos\alpha$.

$$\begin{aligned}(\cos^2\alpha - \sin^2\alpha)\cos\alpha - 2\sin^2\alpha\cos\alpha &= (2\cos^2\alpha - 1)\cos\alpha - 2\cos\alpha(1-\cos^2) \\ &= 4\cos^3\alpha - 3\cos\alpha\end{aligned}$$

Therefore, the following equation is true.

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

□

6 Half Angle Relationships

6.1 List of Relationships

$$\begin{aligned}\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\ \cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\ \tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\ &= \frac{\sin\alpha}{1+\cos\alpha} \\ &= \frac{1-\cos\alpha}{\sin\alpha}\end{aligned}$$

$$\begin{aligned}\sin\theta &= \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \\ \cos\theta &= \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \\ \tan\theta &= \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\end{aligned}$$

6.2 Relationship for $\sin\frac{\alpha}{2}$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Proof. $\frac{\alpha}{2}$ could be substituted instead of α in the proven equation: $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$. $\sin^2\frac{\alpha}{2} = \frac{1}{2}(1-\cos\alpha)$ could be driven.

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

□

6.3 Relationship for $\cos\frac{\alpha}{2}$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Proof. $\frac{\alpha}{2}$ could be substituted instead of α in the proven equation: $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$. $\cos^2\frac{\alpha}{2} = \frac{1}{2}(1+\cos\alpha)$ could be driven.

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

□

6.4 Relationship for $\tan\frac{\alpha}{2}$

$$\begin{aligned}\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\ &= \frac{\sin\alpha}{1+\cos\alpha} \\ &= \frac{1-\cos\alpha}{\sin\alpha}\end{aligned}$$

Proof. $\tan\frac{\alpha}{2}$ could be rewritten as $\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$, which is equal to $\pm\frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}}$. The equation could be further modified.

$$\begin{aligned}\pm\frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \pm\sqrt{\frac{(1-\cos\alpha)(1+\cos\alpha)}{(1+\cos\alpha)(1+\cos\alpha)}} = \pm\sqrt{\frac{\sin^2\alpha}{(1+\cos\alpha)^2}} = \frac{\sin\alpha}{1+\cos\alpha} \\ &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \pm\sqrt{\frac{(1-\cos\alpha)(1-\cos\alpha)}{(1+\cos\alpha)(1-\cos\alpha)}} = \pm\sqrt{\frac{(1-\cos\alpha)^2}{\sin^2\alpha}} = \frac{1-\cos\alpha}{\sin\alpha}\end{aligned}$$

Therefore, the following relationships are true.

$$\begin{aligned}\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\ &= \frac{\sin\alpha}{1+\cos\alpha} \\ &= \frac{1-\cos\alpha}{\sin\alpha}\end{aligned}$$

□

6.5 Relationship for $\sin\theta$

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

Proof. From the double angle relationship $\sin(2\alpha) = \frac{2\tan\alpha}{1+\tan^2\alpha}$, substitute α for 2α.

$$\therefore \sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

□

6.6 Relationship for $\cos\theta$

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

Proof. From the double angle relationship $\cos(2\alpha) = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$, substitute α for 2α.

$$\therefore \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

□

6.7 Relationship for $\tan\theta$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

Proof. From the double angle relationship $\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$, substitute α for 2α.

$$\therefore \tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

□

7 Function Sum and Difference Relationships

7.1 List of Relationships

$$\begin{aligned}
\sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\sin\alpha - \sin\beta &= 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}
\end{aligned}$$

Let $\alpha = \frac{x+y}{2}$ and $\beta = \frac{x-y}{2}$. $\alpha+\beta = x$ and $\alpha-\beta = y$ could be derived.

7.2 Relationship for $\sin\alpha + \sin\beta$

Proof. From $\sin x + \sin y = \sin(\alpha+\beta) + \sin(\alpha-\beta)$, angle sum and difference relationship could be utilized.

$$\begin{aligned}
\sin(\alpha+\beta) + \sin(\alpha-\beta) &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta) + (\sin\alpha\cos\beta - \cos\alpha\sin\beta) \\
&= 2\sin\alpha\cos\beta \\
&= 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}
\end{aligned}$$

Therefore, the following expression is true.

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

□

7.3 Relationship for $\sin\alpha - \sin\beta$

Proof. From $\sin x - \sin y = \sin(\alpha+\beta) - \sin(\alpha-\beta)$, angle sum and difference relationship could be utilized.

$$\begin{aligned}\sin(\alpha+\beta) - \sin(\alpha-\beta) &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta) - (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\\ &= 2\cos\alpha\sin\beta\\ &= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}\end{aligned}$$

Therefore, the following expression is true.

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

□

7.4 Relationship for $\cos\alpha + \cos\beta$

Proof. From $\cos x + \cos y = \cos(\alpha+\beta) + \cos(\alpha-\beta)$, angle sum and difference relationship could be utilized.

$$\begin{aligned}\cos(\alpha+\beta) + \cos(\alpha-\beta) &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + (\cos\alpha\cos\beta + \sin\alpha\sin\beta)\\ &= 2\cos\alpha\cos\beta\\ &= 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}\end{aligned}$$

Therefore, the following expression is true.

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

□

7.5 Relationship for $\cos\alpha - \cos\beta$

Proof. From $\cos x - \cos y = \cos(\alpha+\beta) + \cos(\alpha-\beta)$, angle sum and difference relationship could be utilized.

$$\begin{aligned}\cos(\alpha+\beta) - \cos(\alpha-\beta) &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) - (\cos\alpha\cos\beta + \sin\alpha\sin\beta)\\ &= -2\sin\alpha\sin\beta\\ &= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}\end{aligned}$$

Therefore, the following expression is true.

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

□

8 Function Product Relationships

8.1 List of Relationships

$$\begin{aligned}\sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]\\ \cos\alpha\sin\beta &= \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]\\ \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]\\ \sin\alpha\sin\beta &= -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]\end{aligned}$$

Function sum and difference relationships may be utilized to prove the function product relationships. Let $\alpha = \frac{x+y}{2}$ and $\beta = \frac{x-y}{2}$. $\alpha + \beta = x$ and $\alpha - \beta = y$ could be derived.

8.2 Relationship for $\sin\alpha\cos\beta$

Proof. It is evident that the following equation is true.

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

The equation may be reorganized in terms of α and β.

$$\begin{aligned} \sin x + \sin y &= 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} \\ \sin(\alpha+\beta) + \sin(\alpha-\beta) &= 2\sin\alpha\cos\beta \\ \therefore \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \end{aligned}$$

□

8.3 Relationship for $\cos\alpha\sin\beta$

Proof. It is evident that the following equation is true.

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

The equation may be reorganized in terms of α and β.

$$\begin{aligned} \sin x - \sin y &= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} \\ \sin(\alpha+\beta) - \sin(\alpha-\beta) &= 2\cos\alpha\sin\beta \\ \therefore \cos\alpha\sin\beta &= \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)] \end{aligned}$$

□

8.4 Relationship for $\cos\alpha\cos\beta$

Proof. It is evident that the following equation is true.

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

The equation may be reorganized in terms of α and β.

$$\begin{aligned} \cos x + \cos y &= 2\cos\frac{x+y}{2}\cos\frac{x-y}{2} \\ \cos(\alpha+\beta) + \cos(\alpha-\beta) &= 2\cos\alpha\cos\beta \\ \therefore \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)] \end{aligned}$$

□

8.5 Relationship for $\sin\alpha\sin\beta$

Proof. It is evident that the following equation is true.

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

The equation may be reorganized in terms of α and β.

$$\cos x - \cos y = 2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$
$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = 2\sin\alpha\sin\beta$$
$$\therefore \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$$

□

9 Power Reduction Formulas

9.1 List of Formulas

$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$$
$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

9.2 Formula for $\sin^2\alpha$

Proof. Because $\cos(2\alpha) = 1 - 2\sin^2\alpha$ is proven to be true, the equation could be rearranged to reduce the power for $\sin^2\alpha$.

$$\therefore \sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$$

□

9.3 Formula for $\cos^2\alpha$

Proof. Because $\cos(2\alpha) = 2\cos^2\alpha - 1$ is proven to be true, the equation could be rearranged to reduce the power for $\cos^2\alpha$.

$$\therefore \sin^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

□

10 Example

10.1 Example I: Compute $\cos 10° \cdot \cos 30° \cdot \cos 50° \cdot \cos 70°$.

Solution Using function product relationship and known trigonometric identity, the equation could be rewritten.

$$\begin{aligned}
\cos 10^\circ \cdot \cos 30^\circ \cdot \cos 50^\circ \cdot \cos 70^\circ &= \cos 10^\circ \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}[\cos 50^\circ \cdot \cos 70^\circ] \\
&= \cos 10^\circ \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}[\cos(50^\circ + 70^\circ) + \cos(50^\circ - 70^\circ)] \\
&= \frac{\sqrt{3}}{4} \cos 10^\circ \cdot [\cos 120^\circ + \cos 20^\circ] \\
&= \frac{\sqrt{3}}{4} \cos 10^\circ \cdot [-\frac{1}{2} + \cos 20^\circ] \\
&= -\frac{\sqrt{3}}{8} \cos 10^\circ + \frac{\sqrt{3}}{4} \cos 10^\circ \cos 20^\circ \\
&= -\frac{\sqrt{3}}{8} \cos 10^\circ + \frac{\sqrt{3}}{4} \cdot \frac{1}{2}[\cos 30^\circ + \cos 10^\circ] \\
&= -\frac{\sqrt{3}}{8} \cos 10^\circ + \frac{\sqrt{3}}{4} \cdot \frac{1}{2}[\frac{\sqrt{3}}{2} + \cos 10^\circ] \\
&= -\frac{\sqrt{3}}{8} \cos 10^\circ + \frac{3}{16} + \frac{\sqrt{3}}{8} \cos 10^\circ \\
&= \boxed{\frac{3}{16}}
\end{aligned}$$

□

10.2 Example II: Compute $\sin 18^\circ$.

Solution Let $x = 18^\circ$

$$\begin{aligned}
5x &= 90^\circ \\
2x + 3x &= 90^\circ \\
2x &= 90^\circ - 3x
\end{aligned}$$

Therefore, $\sin 2x = \sin(90^\circ - 3x) = \cos 3x$. In another terms, the following is true.

$$2 \sin x \cos x = 4 \cos^3 x - 3 \cos x$$

Because $\cos x$ is not zero, the following could be yielded.

$$\begin{aligned}
2 \sin x &= 4 \cos^2 x - 3 \\
2 \sin x &= 4(1 - \sin^2 x) - 3 \\
4 \sin^2 x + 2 \sin x - 1 &= 0
\end{aligned}$$

Let $y = \sin x$. Therefore, $4y^2 + 2y - 1 = 0$.

$$y = \frac{-1 \pm \sqrt{5}}{4}$$

However, $y = \frac{-1+\sqrt{5}}{4}$ because $\sin 18^\circ$ is greater than zero.

$$\therefore \sin 18^\circ = \boxed{\frac{-1 + \sqrt{5}}{4}}$$

□