Theorem. *The sum of all natural numbers are* $-\frac{1}{12}$, *or*

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}.$$

Proof. Let S_1, S_2 and S_3 be following sequences.

$$\begin{aligned}
S_1 &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\
S_2 &= 1 - 2 + 3 - 4 + 5 - 6 + \cdots \\
S_3 &= 1 + 2 + 3 + 4 + 5 + 6 + \cdots
\end{aligned}$$

Despite the fact that S_1 is a divergent sequence, the following property may apply.

$$\begin{aligned}
S_1 &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\
S_1 &= \quad\;\; 1 - 1 + 1 - 1 + 1 - \cdots \\
2S_1 &= 1 \\
\therefore S_1 &= \frac{1}{2}
\end{aligned}$$

Similarly, S_2 may be manipulated.

$$\begin{aligned}
S_2 &= 1 - 2 + 3 - 4 + 5 - 6 + \cdots \\
S_2 &= \quad\;\; 1 - 2 + 3 - 4 + 5 - \cdots \\
2S_2 &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\
2S_1 &= S_1 \\
2S_1 &= \frac{1}{2} \\
\therefore S_2 &= \frac{1}{4}
\end{aligned}$$

Lastly, the relationship between S_2 and S_3 may be investigated.

$$\begin{aligned}
S_3 &= 1 + 2 + 3 + 4 + 5 + 6 + \cdots \\
S_2 &= 1 - 2 + 3 - 4 + 5 - 6 + \cdots \\
S_3 - S_2 &= 4 + 8 + 12 + 16 + \cdots \\
&= 4(1 + 2 + 3 + 4 + \cdots) \\
&= 4S_3 \\
3S_3 &= -S_2 \\
3S_3 &= -\frac{1}{4} \\
\therefore S_3 &= -\frac{1}{12}
\end{aligned}$$

□