1965 Hungary Math Olympiad Problem 1

Find integers a, b, and c such that

$$a^2 + b^2 + c^2 + 3 < ab + 3b + 2c.$$

Solution Completing square method could be used to solve the problem.

$$\left(a^2 - ab + \frac{1}{4}b^2\right) + \frac{3}{4}b^2 - 3b + (c-1)^2 + 2 < 0$$
$$\left(a - \frac{1}{2}b\right)^2 + \frac{3}{4}\left(b^2 - 4b + 4\right) - 3 + (c-1)^2 + 2 < 0$$
$$\left(a - \frac{1}{2}b\right)^2 + \frac{3}{4}(b-2)^2 + (c-1)^2 < 1$$

Therefore, the only possible case is when $\boxed{a = 1, b = 2, c = 1}$. □

Problem

Suppose x and y are real numbers. Find the minimum value of

$$u = x^2 + 2xy + 5y^2 + 2x - 6y + 7.$$

Solution Because there exists x^2, y^2, xy, x, y, it is natural to immediately think of the factorization form $(x \pm y + C)^2$.

$$(x + y + 1)^2 + 4y^2 - 8y + 6 = 0$$
$$(x + y + 1)^2 + 4(y - 1)^2 + 2 = 0$$

Thus, the minimum value of u is $\boxed{2}$. □

Problem

Suppose $\frac{1}{2} \leq x^2 + 4y^2 \leq 2$ is true for real numbers x and y.

1. Find the greatest value of $x^2 - 2xy + 4y^2$
2. Find the smallest value of $x^2 - 2xy + 4y^2$

Part 1.

Solution I. Completing square method may provide the maximum value.

$$\begin{aligned} x^2 + 4y^2 &= \frac{2}{3}\left(x^2 - 2xy + 4y^2\right) + \frac{1}{3}\left(x^2 + 4xy + 4y^2\right) \\ &= \frac{2}{3}\left(x^2 - 2xy + 4y^2\right) + \frac{1}{3}(x + 2y)^2 \end{aligned}$$

Continuing,

$$\frac{1}{2} \leq \frac{2}{3}\left(x^2 - 2xy + 4y^2\right) + \frac{1}{3}(x + 2y)^2 \leq 2$$

Notice that when $x = -2y$, $x^2 - 2xy + 4y^2 \leq 3$. However, other cases will lead to $x^2 - 2xy + 4y^2$ being less than or equal to a value smaller than 3. Therefore, $x^2 - 2xy + 4y^2$ will reach the maximum 3 when $x = -2y$, which is the case that provides the maximum value. $\square$

Solution II. Using AM-GM inequality, it is evident that $x^2 + 4y^2 \geq |4xy|$. In other words, $|2xy| \leq 1$. Moreover,

$$\frac{1}{2} - 2xy \leq x^2 - 2xy + 4y^2 \leq 2 - 2xy$$

Consider the case when $x = -1$ and $y = \frac{1}{2}$. $x^2 + 4y^2$ will reach its maximum while xy will face its minimum. Thus, $2 + 1$, or $\boxed{3}$ is the maximum value. $\square$

Part 2.

Solution Completing square method could be used.

$$\begin{aligned} x^2 + 4y^2 &= 2\left(x^2 - 2xy + 4y^2\right) - \left(x^2 - 4xy + 4y^2\right) \\ &= 2\left(x^2 - 2xy + 4y^2\right) - (x - 2y)^2 \end{aligned}$$

Therefore,

$$\frac{1}{2} \leq 2\left(x^2 - 2xy + 4y^2\right) - (x - 2y)^2 \leq 2$$

Notice that $x^2 - 2xy + 4y^2$ will reach its minimum *iff* $x = 2y$. Thus, the minimum value of $x^2 - 2xy + 4y^2$ is $\boxed{\frac{1}{4}}$. $\square$