Rudimentary Knowledge

- For $k = 0, 1, \dots, n-1$, the roots of the four equations could be computed.

$$
\begin{aligned}
x^n &= 1 & x &= e^{\frac{2\pi k}{n} i} \\
x^n &= -1 & x &= e^{\frac{(2k+1)\pi}{n} i} \\
x^n &= i & x &= e^{\frac{\pi + 4\pi k}{2n} i} \\
x^n &= -i & x &= e^{\frac{3\pi + 4\pi k}{2n} i}
\end{aligned}
$$

- Every non-zero complex number could be written in the form of $re^{i\theta}$.
- If $z = \overline{z}$, then z is real.
- $\overline{(z_1 + z_2)^n} = (\overline{z_1} + \overline{z_2})^n$

Problem

Describe all integers n such that the polynomial $x^{2n} + 1 + (x+1)^{2n}$ is divisible by $x^2 + x + 1$.

Solution Let α and β be the roots of the polynomial $x^2 + x + 1$. If $x^{2n} + 1 + (x+1)^{2n}$ is divisible by $x^2 + x + 1$, the following equations must be true.

$$
\begin{aligned}
\alpha^{2n} + 1 + (\alpha + 1)^{2n} &= 0 \\
\beta^{2n} + 1 + (\beta + 1)^{2n} &= 0
\end{aligned}
$$

The equation could be modified with the fact that $\alpha^2 + \alpha + 1 = \beta^2 + \beta + 1 = 0$. However, WLOG, only the case for α could be checked, since the process will be exactly the same for β.

$$
\begin{aligned}
(\alpha + 1)^{2n} = \left(\alpha^2 + 2\alpha + 1\right)^n &= \alpha^n \\
\alpha^{2n} + \alpha^n + 1 &= 0 \\
\frac{\alpha^{3n} - 1}{\alpha^n - 1} &= 0
\end{aligned}
$$

Notice that $\alpha^3 = 1$ In other words, if n is a multiple of 3, a contradiction occurs. However, if n is not a multiple of three, $\alpha^{3n} - 1$ will always be zero, and satisfy the condition. Thus, all integers n are $\boxed{\text{not a multiple of 3}}$.

□

Problem

Let x and y be two k^{th} roots of unity. Prove that $(x + y)^k$ is real.

Proof. Notice that $x\overline{x} = y\overline{y} = 1$ since x and y are k^{th} roots of unity.

$$
(\overline{x} + \overline{y})^k = \left(\frac{1}{x} + \frac{1}{y}\right)^k = \left(\frac{x + y}{xy}\right)^k = (x + y)^k
$$

□

Problem

Show that $\tan n\theta = \frac{\binom{n}{1}\tan\theta - \binom{n}{3}\tan^3\theta + \cdots}{1-\binom{n}{2}\tan^2\theta + \cdots}$.

Proof.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = \binom{n}{0}\cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \cdots + i\left(\binom{n}{1}\cos^{n-1}\theta\sin\theta - \cdots\right)$$

Therefore,

$$\begin{aligned}\tan n\theta &= \frac{\sin n\theta}{\cos n\theta} \\ &= \frac{\binom{n}{1}\cos^{n-1}\theta\sin\theta - \cdots}{\binom{n}{0}\cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \cdots} \\ &= \frac{\binom{n}{1}\tan\theta - \cdots}{1 - \binom{n}{2}\tan^2\theta + \cdots}\end{aligned}$$

□