Problem

Robert writes all positive divisors of the number 216 on separate slips of paper, then places the slips into a hat. He randomly selects three slips from the hat, with replacement. What is the probability that the product of the numbers on the three slips Robert selects is a divisor of 216?

Solution

Key Word Stars and Bars

Notice that $216 = 2^3 \cdot 3^3$. In other words, if $2^{a_1}3^{b_1}$, $2^{a_2}3^{b_2}$ and $2^{a_3}3^{b_3}$ are the three divisors that Robert selects, $a_1 + a_2 + a_3 \leq 3$ and $b_1 + b_2 + b_3 \leq 3$ must be true.

The number of triples (a_1, a_2, a_3) that satisfy the inequality could be solved by adding a constant k where $a_1 + a_2 + a_3 + k = 3$. Because $0 \leq a_i \leq 3$, using Stars and Bars, there would be 3 stars and 3 bars. Thus, there are total of ${}_6C_3 = 20$ triples.

$$\begin{aligned}\text{Probability} &= \frac{20 \cdot 20}{16 \cdot 16 \cdot 16} \\ &= \boxed{\frac{25}{256}}\end{aligned}$$

□

Problem

Find the roots of the polynomial $f(x) = x^8 + x^7 + x^6 + \cdots + x + 1$.

Solution

Key Word Roots of Unity

First, notice that $f(x) = \frac{x^9-1}{x-1}$ if $x \neq 1$. In other words, the roots of $f(x)$ will correspond to the roots of $x^9 = 1$ except at $x = 1$.

$$\begin{aligned}e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{in\theta} &= (\cos\theta + i\sin\theta)^n \\ &= \cos n\theta + i\sin n\theta\end{aligned}$$

With the knowledge above, the following equations are true.

$$\begin{aligned}e^{0\pi} &= (\cos 0 + i\sin 0)^9 = 1 \\ e^{2\pi} &= \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^9 = 1 \\ e^{4\pi} &= \left(\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}\right)^9 = 1 \\ &\vdots \\ e^{16\pi} &= \left(\cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}\right)^9 = 1\end{aligned}$$

In other words, $\boxed{e^{\frac{2\pi}{9}}, e^{\frac{4\pi}{9}}, e^{\frac{6\pi}{9}}, e^{\frac{8\pi}{9}}, e^{\frac{10\pi}{9}}, e^{\frac{12\pi}{9}}, e^{\frac{14\pi}{9}}, e^{\frac{16\pi}{9}}}$ are the roots of the polynomial. □

Problem

Find all solutions to $z^6 + z^4 + z^3 + z^2 + 1 = 0$.

Solution

Key Word Roots of Unity

$$\begin{aligned} z^6 + z^4 + z^3 + z^2 + 1 &= \frac{z^5 - 1}{z - 1} + z^6 - z \\ &= \left(z^5 - 1\right)\left(\frac{1}{z-1} + z\right) \\ &= \left(z^5 - 1\right)\left(\frac{z^2 - z + 1}{z - 1}\right) \\ &= \left(z^5 - 1\right)\left(\frac{\frac{z^3+1}{z+1}}{z - 1}\right) \end{aligned}$$

Therefore, $\boxed{e^{\frac{2\pi}{5}}, e^{\frac{4\pi}{5}}, e^{\frac{6\pi}{5}}, e^{\frac{8\pi}{5}}, e^{\frac{\pi}{3}}, e^{\frac{5\pi}{3}}}$. $\square$