2018 AMC 12B Problem 15

How many odd positive 3-digit integers are divisible by 3 but do not contain the digit 3?

(A) 96 **(B)** 97 **(C)** 98 **(D)** 102 **(E)** 120

Solution

Key Word Case Work

Every integers either have 0, 1, or 2 as the remainder when divided by three. Therefore, the cases could be divided.

$$\begin{cases} \boxed{0}\ \boxed{0}\ \boxed{0} & \Rightarrow 6 \\ \boxed{1}\ \boxed{1}\ \boxed{1} & \Rightarrow 18 \\ \boxed{2}\ \boxed{2}\ \boxed{2} & \Rightarrow 9 \\ \boxed{0}\ \boxed{1}\ \boxed{2} & \Rightarrow 6 \\ \boxed{0}\ \boxed{2}\ \boxed{1} & \Rightarrow 12 \\ \boxed{1}\ \boxed{0}\ \boxed{2} & \Rightarrow 9 \\ \boxed{1}\ \boxed{2}\ \boxed{0} & \Rightarrow 9 \\ \boxed{2}\ \boxed{0}\ \boxed{1} & \Rightarrow 18 \\ \boxed{2}\ \boxed{1}\ \boxed{0} & \Rightarrow 9 \end{cases}$$

Therefore, $6+18+9+6+12+9+9+18+9 = 12+36+12+36 = \boxed{\textbf{(A)}\ 96}$ □

2018 AMC 12B Problem 17

Let p and q be positive integers such that

$$\frac{5}{9} < \frac{p}{q} < \frac{4}{7}$$

and q is as small as possible. What is $q - p$?

(A) 7 **(B)** 11 **(C)** 13 **(D)** 17 **(E)** 19

Solution

Key Property

$$\begin{cases} \frac{a}{b} > 1 & \Rightarrow \frac{a}{b} > \frac{a+1}{b+1} > \frac{a+2}{b+2} > \cdots \\ \frac{a}{b} < 1 & \Rightarrow \frac{a}{b} < \frac{a+1}{b+1} < \frac{a+2}{b+2} < \cdots \end{cases}$$

Using the property above, the following fact could be derived.

$$\frac{35}{63} < \frac{36}{64} < \cdots < \frac{36}{63} \qquad \left(\because \lim_{x \to \infty} \frac{35+x}{63+x} = 1 \right)$$

First, the range for x could be found.

$$\frac{35+x}{63+x} = \frac{36}{63}$$
$$35 \cdot 63 + 63x = 36 \cdot 63 + 36x$$
$$x = \frac{7}{3}$$

In other words, the only possible value for x are 1 and 2. It is evident that q will be smallest when $x = 1$. Therefore, $q - p = 16 - 9 = \boxed{\textbf{(A)}\ 7}$. □

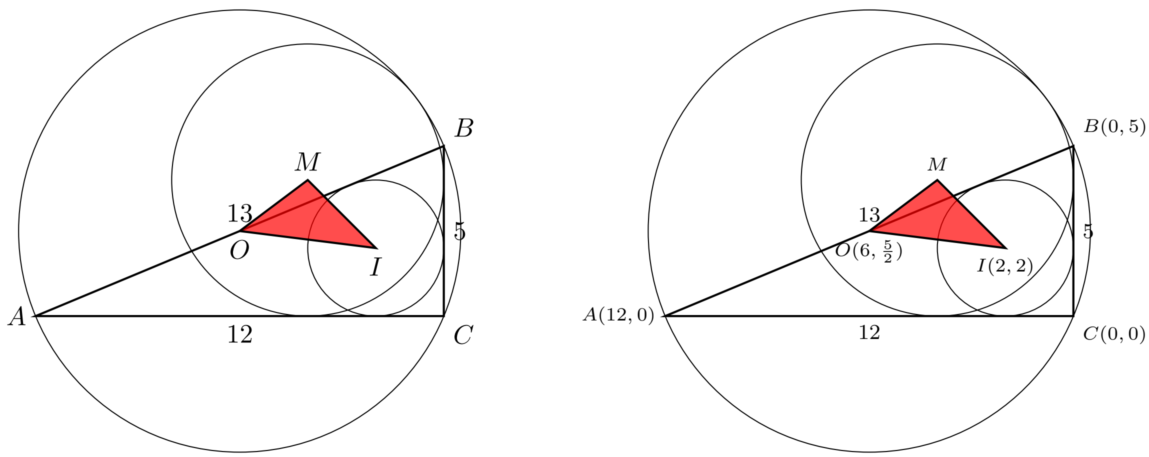
Uploaded a new solution in AOPS!!

2018 AMC 12B Problem 21

In $\triangle ABC$ with side lengths $AB = 13$, $AC = 12$, and $BC = 5$, let O and I denote the circumcenter and incenter, respectively. A circle with center M is tangent to the legs AC and BC and to the circumcircle of $\triangle ABC$. What is the area of $\triangle MOI$?

(A) $\frac{5}{2}$ **(B)** $\frac{11}{4}$ **(C)** 3 **(D)** $\frac{13}{4}$ **(E)** $\frac{7}{2}$

Solution Notice that assigning coordinates to each point may prove new information.



Using the property of incenter, it is evident that the radius of circle I is 2. Moreover, let the coordinate of M be (a, a). Since O is the circumcenter of $\triangle ABC$, the length of OM could be used to find the value of a.

$$\begin{aligned}
\sqrt{(a-6)^2 + \left(a - \frac{5}{2}\right)^2} &= \frac{13}{2} - a \\
2a^2 - 17a + 36 + \frac{25}{4} &= a^2 - 13a + \frac{169}{4} \\
a^2 - 4a &= 0 \\
\therefore a &= 4
\end{aligned}$$

The area of $\triangle IMO$ could be computed using the Shoelace formula.

$$\begin{aligned}
[IMO] &= \frac{1}{2} \begin{vmatrix} 2 & 4 & 6 & 2 \\ 2 & 4 & \frac{5}{2} & 2 \end{vmatrix} \\
&= \frac{1}{2} \left| (8 + 10 + 12) - (8 + 24 + 5) \right| \\
&= \boxed{\textbf{(E) } \frac{7}{2}}
\end{aligned}$$

$\square$