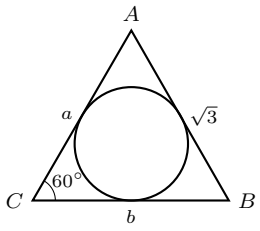
Problem

A Circle O of radius r is inscribed to $\triangle ABC$ at D, E, and F. $\angle C = 60°$. The length of the side opposite to $\angle C$ is $c = \sqrt{3}$. Find the range of r.

Solution



From the diagram, notice that the values of $a + b$ and ab could be written in terms of r.

$$\begin{aligned} a - \tan 60° \cdot r + b - \tan 60° \cdot r &= \sqrt{3} \\ a + b &= \sqrt{3}(2r + 1) \\ \frac{r(a + b + \sqrt{3})}{2} &= \frac{\sqrt{3}ab}{4} \\ \frac{r\left(\sqrt{3}(2r + 1) + \sqrt{3}\right)}{2} &= \frac{\sqrt{3}ab}{4} \\ ab &= 2r(2r + 2) \end{aligned}$$

Let a and b be the root of the equation

$$t^2 - \sqrt{3}(2r + 1) + 4r^2 + 4r = 0$$

Discriminant could be used since a and b are positive numbers.

$$\begin{aligned} 12r^2 + 12r + 3 - 16r^2 - 16r &\geq 0 \\ 4r^2 + 4r - 3 &\leq 0 \\ (2r + 3)(2r - 1) &\leq 0 \\ \therefore \boxed{0 < r \leq \frac{1}{2}} & \end{aligned}$$

□

Problem

Show that there is one and only one of real numbers x, y, and z that is at least $\sqrt[3]{4}$ if $x + y + z = 0$ and $xyz = 1$.

Proof. WLOG, let $x \leq y < z$ and let x and y be the solution to the following equation.

$$t^2 + zt + \frac{1}{z} = 0$$

Using discriminant, $z^2 - \frac{4}{z} \geq 0$. Because z must be a positive number, $z^3 \geq 4$. In other words, only $z \geq \sqrt[3]{4}$. □

Problem

Show that $1 < a + b < \frac{4}{3}$ and $\frac{8}{9} < a^2 + b^2 < 1$ if $a + b + c = a^2 + b^2 + c^2 = 1$, and $a > b > c$.

Proof. Let $x = a - c$ and $y = b - c$. Therefore, $x > y$. Moreover, the following equations are true.

$$\begin{aligned} x + y &= a + b - 2c \\ &= 1 - 3c \\ xy &= (a - c)(b - c) \\ &= ab - c(a + b) + c^2 \\ &= -2c(a + b) + c^2 \\ &= -2c(1 - c) + c^2 \end{aligned}$$

Let x and y be the solution for the following equation.

$$t^2 - (1 - 3c)t - 2c(1 - c) + c^2 = 0$$

Therefore, the following conditions must satisfy.

$$\begin{cases} 1 - 3c > 0 \\ -2c(1 - c) + c^2 > 0 \\ (1 - 3c)^2 + 8c(1 - c) - 4c^2 > 0 \end{cases}$$

Therefore, $c < \frac{1}{3}$, $c < 0$ or $c > \frac{2}{3}$, and $-\frac{1}{3} < c < 1$. In other words, $-\frac{1}{3} < c < 0$.

$$-\frac{1}{3} < 1 - a - b < 0$$
$$\therefore 1 < a + b < \frac{4}{3}$$

$$0 < c^2 < \frac{1}{9}$$
$$0 < 1 - a^2 - b^2 < \frac{1}{9}$$
$$\therefore \frac{8}{9} < a^2 + b^2 < 1$$

□

Problem

Find the range of the function $y = \sqrt{2 - x} + \sqrt{x - 1}$.

Solution I Let $a = \sqrt{2 - x}$ and $b = \sqrt{x - 1}$. Therefore, $a + b = y$ and $(a + b)^2 = 1 + 2\sqrt{(2 - x)(x - 1)} = y^2$. In other words, $ab = \frac{y^2 - 1}{2}$. Let a and b be the roots of the equation $t^2 - yt + \frac{y^2 - 1}{2} = 0$. Using discriminant, $y^2 - 2y^2 + 2 \geq 0$, or $-\sqrt{2} \leq y \leq \sqrt{2}$. However, notice that $y > 0$ and $y \leq -1$ or $y \geq 1$. Therefore, the range for y is $\boxed{[-1, 1]}$. □

Solution II First, notice that $1 \leq x \leq 2$. Moreover, the function is a polynomial. Therefore, the following equations are true.

$$\begin{aligned} y' &= \frac{-1}{\sqrt{2 - x}} + \frac{1}{\sqrt{x - 1}} \\ y'' &= \frac{-1}{2\sqrt{(2 - x)^3}} - \frac{1}{\sqrt{(x - 1)^3}} < 0 \end{aligned}$$

In other words, y is a concave down function that has a critical point at $x = 1.5$. Therefore, the minimum value of y is 1 and the maximum is $\sqrt{2}$. Thus, $\boxed{1 \leq y \leq \sqrt{2}}$. □

Problem

How many distinct ordered triples (x, y, z) satisfy the following equations?

$$\begin{aligned} x + 2y + 4z &= 12 \\ xy + 4yz + 2xz &= 22 \\ xyz &= 6 \end{aligned}$$

Solution First and foremost, notice that the problem might have been more affable if the coefficients of the variables in the second equation were different. If a pattern is found, any equation could be forcibly changed for the pattern to satisfy flawlessly.

$$\begin{aligned} x + 2y + 4z &= 12 \\ 2xy + 8yz + 4xz &= 44 \\ 8xyz &= 48 \end{aligned}$$

Let $a = x$, $b = 2y$, and $c = 4z$. Then, the following equations are true.

$$\begin{aligned} a + b + c &= 12 \\ ab + bc + ca &= 44 \\ abc &= 48 \end{aligned}$$

Using, Vieta's formula, it is evident that a, b, and c are the real roots of the following equation.

$$t^3 - 12t^2 + 44t - 48 = 0$$

Solving for t will lead to the values of a, b, and c.

$$\begin{aligned} t^3 - 12t^2 + 44t - 48 &= 0 \\ (t-2)(t-4)(t-6) &= 0 \\ t &= 2, 4, 6 \end{aligned}$$

Because $2, 4, 6$ are all different numbers, there exists $3!$, or $\boxed{6}$, distinct ordered triples (x, y, z). □

Problem

Find all pairs of integers (a, b) such that the polynomial $ax^{17} + bx^{16} + 1$ is divisible by $x^2 - x - 1$.

Solution I Notice that the coefficients for x^{15} to x are all zero for the polynomial. Therefore, the polynomial could intuitively be factored.

$$(x^2 - x - 1)(\underbrace{\qquad\qquad\qquad\qquad}_{} + x - 1)$$

Notice that a Fibonacci sequence with alternating signs must appear for the coefficients of x^i. Notice that $F_{15} = 987$ and $F_{14} = 610$. Therefore, the pair of integer is $\boxed{(987, -1597)}$. □

Solution II Let α and β be the roots of the divisor. Notice that α and β are also the roots of the first polynomial. Therefore, the following equations are true.

$$\begin{aligned} a\alpha^{17} + b\alpha^{16} &= -1 \\ a\alpha^{17}\beta^{16} + b\alpha^{16}\beta^{16} &= -\beta^{16} \\ a\beta^{17}\alpha^{16} + b\alpha^{16}\beta^{16} &= -\alpha^{16} \\ a\alpha + b &= -\beta^{16} \\ a\beta + b &= -\alpha^{16} \end{aligned}$$

Therefore, $a(\alpha - \beta) = \alpha^{16} - \beta^{16} = (\alpha^8 + \beta^8)(\alpha^4 + \beta^4)(\alpha^2 + \beta^2)(\alpha + \beta) = 987$. Moreover, $b = -1597$. Thus, the ordered pair is $\boxed{(987, -1597)}$.