Problem

Let $x = \sqrt{a^2+a+1} - \sqrt{a^2-a+1}$, where a is real. Find all possible values of x.

Solution Notice that because a is a real number, discriminant could be used to show that $\sqrt{a^2+a+1} + \sqrt{a^2-a+1}$ is never zero. Thus, x could be rewritten.

$$x = \frac{2a}{\sqrt{a^2+a+1} + \sqrt{a^2-a+1}}$$

Notice that when a is greater than zero, the denominator is greater than a, leading to $x < 2$. When a is zero, x is simply zero. Furthermore, if a is less than zero, then x is merely a negative number. In other words, $x < 2$ for all values of a.

$$\begin{aligned} x^2 + 2x\sqrt{a^2-a+1} + a^2 - a + 1 &= a^2 + a + 1 \\ 4x^2\left(a^2 - a + 1\right) &= 4a^2 - 4ax^2 + x^4 \\ 4x^2a^2 - 4a^2 &= x^4 - 4x^2 \\ a^2 &= \frac{x^4 - 4x^2}{4x^2 - 4} = \frac{x^2(x^2-4)}{4(x^2-1)} \geq 0 \end{aligned}$$

Notice that $\frac{x^2(x^2-4)}{4(x^2-1)}$ is positive *iff* $x^2(x^2-4)(x^2-1)$ is positive. Meaning, x^2 is either in the interval $[0,1]$ or $[4,\infty)$. However, the second interval is not applicable. Also, $\lim_{a\to\infty} \sqrt{a^2+a+1} - \sqrt{a^2-a+1} = 1$ and $\lim_{a\to-\infty} \sqrt{a^2+a+1} - \sqrt{a^2-a+1} = -1$. Therefore, $\boxed{-1 < x < 1}$. $\square$

Problem

Solve the equation $x^3 - 3x = \sqrt{x+2}$ for all real values of x.

Solution Notice that when $x = 2$, the equation satisfies. Therefore, intuitively, the cases for x could be set.

$$\begin{cases} x \leq 2 \implies \sqrt{x+2} \geq x \\ x > 2 \implies \sqrt{x+2} < x \end{cases}$$

Observe the second equation.

$$\begin{aligned} x^3 - 3x - \sqrt{x+2} &= 0 \\ x^3 - 3x - x &< 0 \\ x(x^2-4) &< 0 \\ \therefore x < -2 \text{ or } 0 < x &< 2 \end{aligned}$$

Due to a contradiction, the fact that the roots of the equation are never greater than 2 could be deduced. Moreover, since x is a real number, $-2 \leq x \leq 2$.

GENERAL RULE OF THUMB: IF YOU SEE $-a \leq x \leq a$, IMMEDIATELY THINK OF TRIGONOMETRY!!!!

Let $x = 2\cos\theta$. Therefore, the following equations could be deduced.

$$\begin{aligned} 8\cos^3\theta - 6\theta &= \sqrt{2\cos\theta + 2} \\ \cos 3\theta &= \sqrt{\frac{\cos\theta + 1}{2}} \\ \cos 3\theta &= \cos\frac{\theta}{2} \\ \therefore x &= \boxed{2, 2\cos\frac{4\pi}{5}, 2\cos\frac{4\pi}{7}} \end{aligned}$$

$\square$