2018 AMC 12A Problem 22

The solutions to the equations $z^2 = 4 + 4\sqrt{15}i$ and $z^2 = 2 + 2\sqrt{3}i$, where $i = \sqrt{-1}$, form the vertices of a parallelogram in the complex plane. The area of this parallelogram can be written in the form $p\sqrt{q} - r\sqrt{s}$, where p, q, r, and s are positive integers and neither q nor s is divisible by the square of any prime number. What is $p + q + r + s$?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

Solution

Key Words Shoelace Formula

Notice the following equalities.

$$\begin{aligned} 4 + 4\sqrt{15}i &= 4 + 2\sqrt{60}i \\ &= \left(\sqrt{10} + \sqrt{6}i\right)^2 \end{aligned}$$

In other words, the two roots for z in the first equation are $\pm\sqrt{10} \pm \sqrt{6}i$.

Similarly,

$$2 + 2\sqrt{3}i = \left(\sqrt{3} + i\right)^2$$

Thus, the two roots for z in the first equation are $\pm\sqrt{3} \pm i$.

Using the shoelace formula, the area of the parallelogram could be computed.

$$\begin{aligned} \text{Area} &= \frac{1}{2}\begin{vmatrix} \sqrt{10} & -\sqrt{3} & -\sqrt{10} & \sqrt{3} & \sqrt{10} \\ \sqrt{6} & -1 & -\sqrt{6} & 1 & \sqrt{6} \end{vmatrix} \\ &= \frac{1}{2}\left\{ \left(-\sqrt{10} + \sqrt{18} - \sqrt{10} + \sqrt{18}\right) - \left(-\sqrt{18} + \sqrt{10} - \sqrt{18} + \sqrt{10}\right) \right\} \\ &= 2\sqrt{18} - 2\sqrt{10} \\ &= 6\sqrt{2} - 2\sqrt{10} \end{aligned}$$

Therefore, $6 + 2 + 2 + 10 = \boxed{\textbf{(A) } 20}$. □

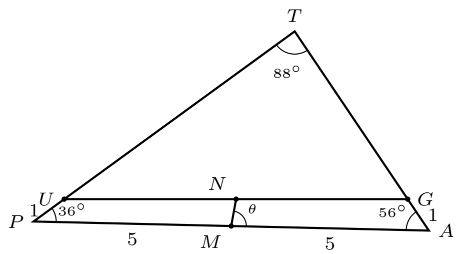
2018 AMC 12A Problem 23

In $\triangle PAT$, $\angle P = 36^\circ$, $\angle A = 56^\circ$, and $PA = 10$. Points U and G lie on sides $\overline{TP}$ and $\overline{TA}$, respectively, so that $PU = AG = 1$. Let M and N be the midpoints of segments $\overline{PA}$ and $\overline{UG}$, respectively. What is the degree measure of the acute angle formed by lines MN and PA?

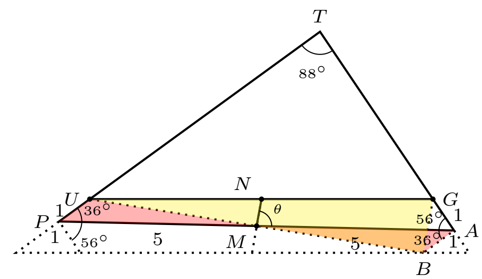
(A) 76 **(B)** 77 **(C)** 78 **(D)** 79 **(E)** 80

Solution I.

First and foremost, draw the diagram.



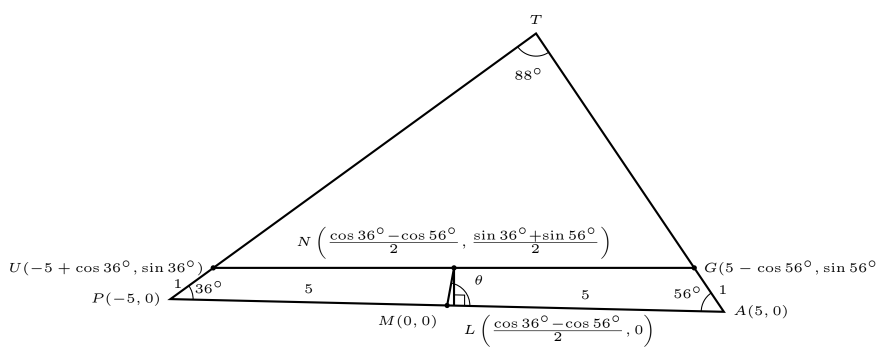
In order to use the angles 36° and 56° and midpoints, the quadrilateral could be attached through rotation.



Notice that $\triangle UMN$ is similar to $\triangle UBG$. In other words, MN is parallel to GB. Therefore, $\theta = 180^\circ - 56^\circ - 44^\circ = \boxed{\textbf{(E)}\ 80}$. $\square$

Solution II.

Assigning coordinate values to few points may assist in problem solving.



Consider the right triangle NML. It is evident that $\tan\theta = \frac{\frac{\sin 36^\circ + \sin 56^\circ}{2}}{\frac{\cos 36^\circ - \cos 56^\circ}{2}}$.

$$\begin{aligned} \tan\theta &= \frac{\sin 36^\circ + \sin 56^\circ}{\cos 36^\circ - \cos 56^\circ} \\ &= \frac{2\sin 46^\circ \cos 10^\circ}{2\sin 46^\circ \sin 10^\circ} \\ &= \frac{\sin 80^\circ}{\cos 80^\circ} \\ &= \tan 80^\circ \end{aligned}$$

Because θ is an acute angle, the measure of the angle must be $\boxed{\textbf{(E)}\ 80}$.