Problem

Determine the sum of the angles A an B, where $0° \leq A, B \leq 180°$, and

$$\sin A + \sin B = \sqrt{\frac{3}{2}}, \quad \cos A + \cos B = \frac{1}{2}.$$

Solution

$$\frac{2 \cdot \sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)}{2 \cdot \cos\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)} = \frac{\sqrt{\frac{3}{2}}}{\sqrt{\frac{1}{2}}}$$

$$\tan\left(\frac{A+B}{2}\right) = \sqrt{3}$$

Therefore, $A + B = \boxed{120°}$. $\hfill\square$

1986 AIME Problem 3

If $\tan x + \tan y = 25$ and $\cot x + \cot y = 30$, then what is $\tan(x + y)$?

Solution Let $a = \tan x$ and $b = \tan y$. Therefore, $a + b = 25$ and $\frac{1}{a} + \frac{1}{b} = 30$. Thus, $ab = \frac{5}{6}$. Because $\tan(x + y) = \frac{a+b}{1-ab}$, the final answer is $\boxed{150}$. $\hfill\square$

Problem

Let A and B be acute angles such that $\tan A = \frac{1}{7}$ and $\sin B = \frac{1}{\sqrt{10}}$. Find the degree measure of $A + 2B$ without using a calculator.

Solution Because A and B are acute angles, consider a right triangle with sides lengths 1-7-$\sqrt{50}$ for A and 1-3-$\sqrt{10}$ for B.

$$\sin(A + 2B) = \sin A(\cos^2 B - \sin^2 B) + 2\sin B \cos B \cos A$$

$$= \frac{1}{\sqrt{50}}\left(\frac{9}{10} - \frac{1}{10}\right) + 2 \cdot \frac{1}{\sqrt{10}} \cdot \frac{3}{\sqrt{10}} \cdot \frac{7}{\sqrt{50}}$$

$$= \frac{50}{10\sqrt{50}}$$

$$= \frac{\sqrt{2}}{2}$$

Therefore $A + 2B = \boxed{45°}$ since both A and B are less than $45°$. $\hfill\square$

Problem

Find all acute angles θ such that $\sin\theta + \sin 2\theta = \cos\theta + \cos 2\theta$.

Solution I.

$$2 \cdot \sin\frac{3\theta}{2} \cdot \cos\frac{\theta}{2} = 2 \cdot \cos\frac{3\theta}{2} \cdot \cos\frac{\theta}{2}$$

$$\tan\frac{3\theta}{2} = 1$$

$$\therefore \theta = \boxed{30°}$$

Solution II.

$$(\sin\theta - \cos\theta)^2 = (\cos 2\theta - \sin 2\theta)^2$$

$$1 - 2\sin\theta\cos\theta = 1 - 2\cos 2\theta\sin 2\theta$$

$$\sin 2\theta = 2\cos 2\theta \sin 2\theta$$

$$\cos 2\theta = \frac{1}{2}$$

$$\theta = \boxed{30°}$$

Problem

Solve the system

$$2x + yx^2 = y,$$
$$2y + zy^2 = z,$$
$$2z + xz^2 = x$$

for real numbers x, y, and z.

Solution Because x can be any number, let $x = \tan\theta$. Notice that $y = \frac{2x}{1-x^2}$. In other words, $y = \tan 2\theta$. Similarly, $z = \frac{2y}{1-y^2}$. In simpler terms, $z = \tan 4\theta$. Continuing, $x = \frac{2z}{1-z^2}$. Namely, $x = \tan\theta = \tan 8\theta$.

The fact that $8\theta - \theta = \pi n$ for $n \in \mathbb{Z}$ could be derived. Since $\theta = \frac{\pi n}{7}$, substitution could be used.

$$x = \tan\frac{\pi n}{7}$$
$$y = \tan\frac{2\pi n}{7}$$
$$z = \tan\frac{4\pi n}{7}$$

Because tan functions have a period of π, it is evident that there would be 7 $0 \sim 6$ possible answers since for $n \geq 7$, the values will repeat. Thus, the solutions for the system are $\boxed{(0,0,0), \left(\tan\frac{\pi}{7}, \tan\frac{2\pi}{7}, \tan\frac{4\pi}{7}\right),}$

$\boxed{\left(\tan\frac{2\pi}{7}, \tan\frac{4\pi}{7}, \tan\frac{8\pi}{7}\right), \left(\tan\frac{3\pi}{7}, \tan\frac{6\pi}{7}, \tan\frac{12\pi}{7}\right), \left(\tan\frac{4\pi}{7}, \tan\frac{8\pi}{7}, \tan\frac{16\pi}{7}\right), \left(\tan\frac{5\pi}{7}, \tan\frac{10\pi}{7}, \tan\frac{20\pi}{7}\right),}$

$\boxed{\left(\tan\frac{6\pi}{7}, \tan\frac{12\pi}{7}, \tan\frac{24\pi}{7}\right)}$. □

1995 AIME Problem 7

Given that

$$(1 + \sin t)(1 + \cos t) = \frac{5}{4},$$

find $(1 - \sin t)(1 - \cos t)$.

Solution

$$\begin{cases} 1 + \sin t + \cos t + \sin t \cdot \cos t = \frac{5}{4} \\ 1 - \sin t - \cos t + \sin t \cdot \cos t = x \end{cases}$$

Therefore, utilizing both equations will lead to following equations.

$$1 + \sin t \cdot \cos t = \frac{5}{8} + \frac{x}{2}$$
$$\sin t + \cos t = \frac{5}{8} - \frac{x}{2}$$

Therefore,

$$
1 + 2\sin t \cos t = \left(\frac{5}{8} - \frac{x}{2} \right)^2
$$

$$
\sin t \cdot \cos t = \frac{\left(\frac{5}{8} - \frac{x}{2} \right)^2 - 1}{2}
$$

$$
1 + \frac{\left(\frac{5}{8} - \frac{x}{2} \right)^2 - 1}{2} = \frac{5}{8} + \frac{x}{2}
$$

$$
128 + 25 - 40x + 16x^2 - 64 = 80 + 64x
$$

$$
16x^2 - 104x + 9 = 0
$$

$$
\therefore x = \boxed{\frac{13 - 4\sqrt{10}}{4}}
$$

$\square$

Problem

Show that $\frac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = \tan 3x$ for all x for which $\tan 3x$ is defined.

Proof.

$$
\frac{2\sin 3x \cos 2x + \sin 3x}{2\cos 3x \cos 2x + \cos 3x} = \tan 3x \cdot \left(\frac{2\cos 2x + 1}{2\cos 2x + 1} \right)
$$

$$
= \tan 3x
$$

$\square$