1996 AIME Problem 10

Find the smallest positive integer solution to $\tan 19x^\circ = \frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}$.

Solution I.

Solution

$$\begin{aligned}\frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ} &= \frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ} \cdot \frac{\frac{1}{\cos 96^\circ}}{\frac{1}{\cos 96^\circ}} \\ &= \frac{1 + \tan 96^\circ}{1 - \tan 96^\circ} \\ &= \frac{\tan 45^\circ + \tan 96^\circ}{1 - \tan 45^\circ \tan 96^\circ} \\ &= \tan(45+96)^\circ \\ &= \tan 141^\circ\end{aligned}$$

Therefore, $\tan 19x^\circ = \tan 141^\circ$. In other words, $19x = 141 + 180k$ for some integer k.

$$\begin{aligned}141 + 180k &\equiv 0 \pmod{19} \\ 8 + 9k &\equiv 0 \pmod{19} \\ 9k &\equiv 11 \pmod{19} \\ 9k &\equiv 11, 30, 49, 68, 87, 106, 125, 144 \pmod{19}\end{aligned}$$

Substituting k, the smallest value of $19x$ is $141 + 180 \cdot 16$, which is $\boxed{159}$. □

Solution II.

Solution Notice that $\sin 45^\circ = \cos 45^\circ$. Therefore, all terms in the numerator and the denominator could be multiplied by either $\sin 45^\circ$ or $\cos 45^\circ$.

$$\begin{aligned}\frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ} &= \frac{\cos 96^\circ \sin 45^\circ + \sin 96^\circ \cos 45^\circ}{\cos 96^\circ \cos 45^\circ - \sin 96^\circ \sin 45^\circ} \\ &= \frac{\sin(96+45)^\circ}{\cos(96+45)^\circ} \\ &= \tan 141^\circ\end{aligned}$$

Modular arithmetic from above could be used to obtain answer $\boxed{159}$. □

Uploaded a new solution in AOPS!!

2008 AIME II Problem 8

Let $a = \frac{\pi}{2008}$. Find the smallest positive integer n such that

$$2[\cos a \cdot \sin a + \cos 4a \cdot \sin 2a + \cos 9a \cdot \sin 3a + \ldots \cos(n^2a) \cdot \sin(na)]$$

is an integer.

Solution

Key Word Telescoping

Recall that $\cos\alpha \sin\beta = \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}$.

$$\begin{aligned}
&2[\cos a \cdot \sin a + \cos 4a \cdot \sin 2a + \cos 9a \cdot \sin 3a + \ldots \cos(n^2a) \cdot \sin(na)]\\
&= 2 \cdot \left[\frac{\sin(a+a)-\sin(a-a)}{2} + \frac{\sin(4a+2a)-\sin(4a-2a)}{2} + \cdots + \frac{\sin(n^2a+na)-\sin(n^2a-na)}{2}\right]\\
&= (\sin(a+a)-\sin(a-a)) + (\sin(4a+2a)-\sin(4a-2a)) + \cdots + (\sin(n^2a+na)-\sin(n^2a-na))\\
&= (\sin 2a - \sin 0a) + (\sin 6a - \sin 2a) + \cdots + (\sin(n^2a+na)-\sin(n^2a-na))\\
&= (\sin 2a + \sin 6a + \sin 12a + \cdots + \sin(n^2a+na)) - (\sin 0a + \sin 2a + \sin 6a + \cdots + \sin(n^2a-na))\\
&= -\sin 0 + \sin(n^2a+na)\\
&= \sin(n+1)na\\
&= \sin\frac{n(n+1)\pi}{2008}
\end{aligned}$$

Using the properties of sine functions, notice that $\frac{n(n+1)\pi}{2008}$ must be in the form of $\frac{k\pi}{2}$ for some integer k.

$$\therefore \frac{n(n+1)}{1004} = k$$

Notice that $1004 = 2^2 \cdot 251$. n cannot be 250. Therefore, the smallest positive integer n is $\boxed{251}$. □

1989 AHSME Problem 28

Find the sum of the roots of $\tan^2 x - 9\tan x + 1 = 0$ that are between $x = 0$ and $x = 2\pi$ radians.

Solution Multiply $\cos^2 x$ on both sides.

$$\begin{aligned}
\sin^2 x - 9\sin x\cos x + \cos^2 x &= 0\\
1 - 9\sin x\cos x &= 0\\
\sin x\cos x &= \frac{1}{4}\\
\sin 2x &= \frac{2}{9}
\end{aligned}$$

The possible values for $2x$ are $\alpha, \pi-\alpha, 2\pi+\alpha, 3\pi-\alpha$. Therefore, the sum of all roots are $\boxed{3\pi}$ □