2019 AMC 12A Problem 19

In $\triangle ABC$ with integer side lengths,

$$\cos A = \frac{11}{16}, \qquad \cos B = \frac{7}{8}, \qquad \text{and} \qquad \cos C = -\frac{1}{4}.$$

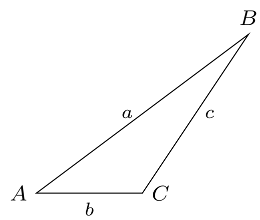
What is the least possible perimeter for $\triangle ABC$?

Solution

Key Word Law of Sines

YOU SHOULD BE FLEXIBLE ENOUGH TO THINK OF sin **WHEN YOU SEE** cos**!!!!**

$$\begin{aligned}
\cos A = \frac{11}{16} &\implies \sin A = \frac{3\sqrt{15}}{16} \\
\cos B = \frac{7}{8} &\implies \sin B = \frac{\sqrt{15}}{8} \\
\cos C = -\frac{1}{4} &\implies \sin C = \frac{\sqrt{15}}{4}
\end{aligned}$$



Therefore, using the law of sines, we could infer that the following expressions are true.

$$\begin{aligned}
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} &\Rightarrow \frac{a}{\frac{3\sqrt{15}}{16}} = \frac{b}{\frac{\sqrt{15}}{8}} = \frac{c}{\frac{\sqrt{15}}{4}} \\
&\Rightarrow \frac{a}{3} = \frac{b}{2} = \frac{c}{4} \\
&\therefore \Rightarrow a : b : c = 3 : 2 : 4
\end{aligned}$$

Because $\triangle ABC$ has integer side lengths, $3 + 2 + 4 = \boxed{9}$. $\square$

2014 AMC 12B Problem 25

Find the sum of all the positive solutions of

$$2\cos(2x)\left(\cos(2x)-\cos\left(\frac{2014\pi^2}{x}\right)\right)=\cos(4x)-1$$

Solution

Key Word Trigonometric Identities

First and foremost, the equation above could be simplified using trigonometric identities.

$$\begin{aligned}
2\cos(2x)\left(\cos(2x)-\cos\left(\frac{2014\pi^2}{x}\right)\right)&=\cos(4x)-1\\
2\cos(2x)\left(\cos(2x)-\cos\left(\frac{2014\pi^2}{x}\right)\right)&=(2\cos^2(2x)-1)-1\\
\cos(2x)\left(\cos(2x)-\cos\left(\frac{2014\pi^2}{x}\right)\right)&=\cos^2(2x)-1\\
\cos(2x)\cos\left(\frac{2014\pi^2}{x}\right)&=1
\end{aligned}$$

Because cos functions are bounded by the interval $[-1,1]$, $\cos(2x)=1$ and $\cos\left(\frac{2014\pi^2}{x}\right)=1$. In other words, $2x=2n\pi$ and $\frac{2014\pi^2}{x}=2n'\pi$ where $\{n,n'\}\in\mathbb{Z}$.

$$\begin{aligned}
2x&=2n\pi\\
x&=n\pi\\
\frac{2014\pi^2}{x}&=2n'\pi\\
\frac{1007\pi}{x}&=n'\\
\therefore x&=\frac{1007\pi}{n'}
\end{aligned}$$

$$\begin{aligned}
x&\Rightarrow \pi, 19\pi, 53\pi, 1007\pi\\
&\Rightarrow \pi+19\pi+53\pi+1007\pi=\boxed{1080\pi}
\end{aligned}$$

□