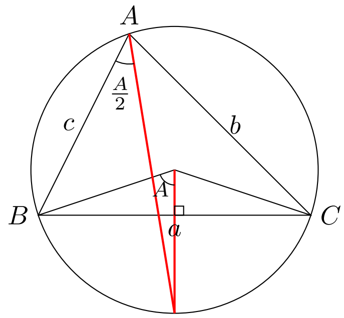
Problem

Show that in $\triangle ABC$, we have $4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{R}$.

Possible Trigger



Unfortunately, the trigger cannot be utilize since not much of the lengths are known.

Proof. Half Angle Formula and the Law of Cosines could be utilized.

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos\frac{A}{2} = \sqrt{\frac{1+\frac{b^2+c^2-a^2}{2bc}}{2}} = \sqrt{\frac{b^2+2bc+c^2-a^2}{4bc}}$$

$$= \sqrt{\frac{(b+c)^2-a^2}{4bc}} = \sqrt{\frac{(a+b+c)(-a+b+c)}{4bc}}$$

$$= \sqrt{\frac{s(s-a)}{bc}}$$

The form triggers area formulas such as Heron's Formula and area using the radius of circumcircle.

$$4\cdot\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ac}}\cdot\sqrt{\frac{s(s-c)}{ab}} = 4\cdot\sqrt{\frac{s^3(s-a)(s-b)(s-c)}{a^2b^2c^2}}$$

$$= \frac{4s}{abc}\cdot\sqrt{s(s-a)(s-b)(s-c)} = \frac{4s}{abc}\cdot\frac{abc}{4R}$$

$$= \frac{s}{R}$$

□

Problem

In $\triangle ABC$, the angles A, B, and C satisfy the equation $\cos A\cos B + \sin A\sin B\sin C = 1$. Determine all possible values of $\angle C$.

Solution $\cos A\cos B + \sin A\sin B$ reminds me of $\cos(A-B)$.

$$\cos(A-B) - \sin A\sin B + \sin A\sin B\sin C = 1$$

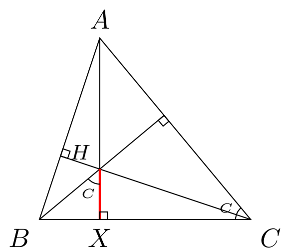
$$\cos(A-B) = 1 + \sin A\sin B(1-\sin C)$$

Notice that the left hand side is in the interval $[-1,1]$. In other words, because $\sin A$ and $\sin B$ are always positive, $\sin C = 1$. The only possible case is when $\boxed{m\angle C = 90^\circ}$. □

Problem

Let $\triangle ABC$ be an acute triangle whose altitudes meet at point H, and let X be on $\overline{BC}$ such that $\overline{AX} \perp \overline{BC}$. Show that $HX = 2R\cos B\cos C$.

Proof. First and foremost, WLOG, the diagram could be drawn.



Using trigonometric identities and law of sines, the following equations could be proven.

$$\begin{aligned} HX = BX \cdot \frac{1}{\tan C} &= BX \cdot \cot C \\ &= AB \cdot \cos B \cdot \frac{\cos C}{\sin C} \\ &= \frac{AB}{\sin C} \cdot \cos B \cos C \\ &= 2R\cos B\cos C \end{aligned}$$

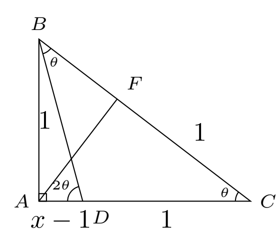
□

Problem

In triangle ABC, D is on $\overline{AC}$ and F is on $\overline{BC}$. Also, $\overline{AB} \perp \overline{AC}$, $\overline{AF} \perp \overline{BC}$, and $BD = DC = FC = 1$. Find $\overline{AC}$.

Solution

Similar triangle could be used to solve the problem. However, because methods with similar triangles are way too common, let's try something new like using trigonometric identities.



Trigonometric identities could be utilized.

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\frac{x-1}{1} = 2 \cdot \left(\frac{1}{x}\right)^2 - 1$$

$$x^3 - x^2 = 2 - x^2$$

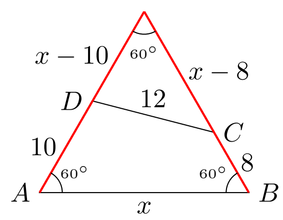
$$\therefore x = \boxed{\sqrt[3]{2}} \quad (\because x \in \mathbb{R})$$

□

2005 AIME I Problem 7

In quadrilateral $ABCD$, $BC = 8$, $CD = 12$, $AD = 10$, and $\angle A = \angle B = 60^\circ$. Find AB.

Solution



The law of cosines could be used.

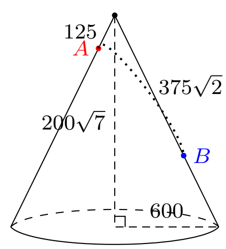
$$\cos 60^\circ = \frac{(x-10)^2 + (x-8)^2 - 12^2}{2(x-10)(x-8)}$$
$$(x-10)(x-8) = (x-10)^2 + (x-8)^2 - 12^2$$
$$\therefore x = \boxed{9 + \sqrt{141}}$$

□

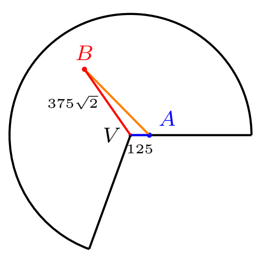
2004 AIME II Problem 11

A right circular cone has a base with radius 600 and height $200\sqrt{7}$. An ant starts at a point on the surface of the cone whose distance from the vertex of the cone is 125, and crawls along the surface of the cone to a point on the exact opposite side of the cone whose distance from the vertex is $375\sqrt{2}$. Find the least distance that the ant could have crawled.

Solution



To find the shortest length from the red to blue points, the net of the side of the cone could be drawn.



The angle YVX is equal to $360^\circ \cdot \frac{1200\pi}{1600\pi} \cdot \frac{1}{2}$, or 135°. Therefore, the law of cosines could be utilized.

$$AB = \sqrt{(375\sqrt{2})^2 + 125^2 - 2 \cdot (375\sqrt{2})(125)(\cos 135^\circ)} = \boxed{625}$$

□

Uploaded a new solution in AOPS!!