Problem

The sequence $\{a_n\}$ satisfies $a_0 = 0$ and $a_{n+1} = \frac{8}{5}a_n + \frac{6}{5}\sqrt{4^n - {a_n}^2}$ for $n \geq 0$. Find a_{10}.

Solution

Key Word $a\cos x + b\sin x = R\cos(x-\alpha)$ where $R = \sqrt{a^2+b^2}$ and $\tan\alpha = \frac{b}{a}$, Trigonometric Identities

Let $b_n = \frac{a_n}{2^n}$, $b_{n+1} = \frac{a_{n+1}}{2^{n+1}}$, and ${b_n}^2 = \frac{{a_n}^2}{4^n}$. Therefore, the following equations are true.

$$\begin{aligned} 2^{n+1} \cdot b_{n+1} &= \frac{8}{5} \cdot 2^n \cdot b_n + \frac{6}{5}\sqrt{4^n - 4^n \cdot {b_n}^2} \\ 2b_{n+1} &= \frac{8}{5} \cdot b_n + \frac{6}{5}\sqrt{1 - {b_n}^2} \\ b_{n+1} &= \frac{4}{5} \cdot b_n + \frac{3}{5}\sqrt{1 - {b_n}^2} \end{aligned}$$

Lemma. $\forall\ i \in \mathbb{Z}, a_i \in \mathbb{R}$.

Proof. Inductive steps may lead to the complete proof of the lemma.

Base Case

$a_0 = 0$ is given by the problem. Through substitution, it could be inferred that $a_1 = \frac{6}{5}$, which is a real number. Thus, the base case holds true.

Inductive Steps

Hypothesis. *If* $a_{n+1} = \frac{8}{5}a_n + \frac{6}{5}\sqrt{4^n - {a_n}^2}$ *is a real number, then* $a_{n+2} = \frac{8}{5}a_{n+1} + \frac{6}{5}\sqrt{4^{n+1} - {a_{n+1}}^2}$ *is also a real number.*

Proof.

$$\begin{aligned} a_{n+2} &= \frac{8}{5}a_{n+1} + \frac{6}{5}\sqrt{4^{n+1} - {a_{n+1}}^2} \\ &= \frac{8}{5}\left(\frac{8}{5}a_n + \frac{6}{5}\sqrt{4^n - {a_n}^2}\right) + \frac{6}{5}\sqrt{4^{n+1} - \left(\frac{8}{5}a_n + \frac{6}{5}\sqrt{4^n - {a_n}^2}\right)^2} \end{aligned}$$

Because the first term is always real, only the second term could be investigated.

$$\frac{6}{5}\sqrt{4^{n+1} - \left(\frac{8}{5}a_n + \frac{6}{5}\sqrt{4^n - {a_n}^2}\right)^2}$$

The sign of the expression inside the square root could be checked.

$$\begin{aligned} &4^{n+1} - \left(\frac{8}{5}a_n + \frac{6}{5}\sqrt{4^n - {a_n}^2}\right)^2 \\ &= 4^{n+1} - \left(\frac{64}{25}{a_n}^2 + \frac{96}{25}a_n\sqrt{4^n - {a_n}^2} + \frac{36}{25}(4^n - {a_n}^2)\right) \\ &= \frac{64}{25} \cdot 4^n - \frac{28}{25}{a_n}^2 - \frac{96}{25}a_n\sqrt{4^n - {a_n}^2} \end{aligned}$$

From the hypothesis, the range for a_n could be found.

$$-2^n \leq a_n \leq 2^n$$

Let $a_n = x$ and $2^n = y$.

$$\begin{aligned} &\Rightarrow \frac{64}{25}y^2 - \frac{28}{25}x^2 - \frac{96}{25}x\sqrt{y^2 - x^2} \quad \text{where } -y \leq x \leq y \\ &\Rightarrow 16y^2 - 7x^2 - 24x\sqrt{y^2 - x^2} \end{aligned}$$

Notice that $-1 \leq \frac{x}{y} \leq 1$. Therefore, let $\frac{x}{y} = \sin\theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. In other words, $x = y\sin\theta$

$$\begin{aligned}
&16y^2 - 7(y\sin\theta)^2 - 24(y\sin\theta)\sqrt{y^2 - (y\sin\theta)^2} \\
&= 16y^2 - 7(y\sin\theta)^2 - 24(y\sin\theta)\sqrt{y^2 - (y\sin\theta)^2} \\
&= 16y^2 - 7y^2\sin^2\theta - 24y^2\sin\theta\cos\theta \\
&= y^2(16 - 7\sin^2\theta - 24\sin\theta\cos\theta)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow 16 - 7\sin^2\theta - 24\sin\theta\cos\theta \\
&= 16 - 7\sin^2\theta - 12\sin 2\theta \\
&= 16 - \frac{7 - 7\cos 2\theta}{2} - 12\sin 2\theta \\
&= \frac{25}{2} + \frac{7\cos 2\theta}{2} - 12\sin 2\theta \ (-\pi \leq 2\theta \leq \pi)
\end{aligned}$$

Use the fact that $a\cos x + b\sin x = R\cos(x - \alpha)$ where $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = \frac{b}{a}$ could be utilized.

$$\begin{aligned}
&\frac{25}{2} + \frac{7\cos 2\theta}{2} - 12\sin 2\theta \\
&= \frac{25}{2} + \left(\sqrt{\frac{49}{4} + 144}\right)\cos(2\theta - \alpha) \\
&= \frac{25}{2} + \frac{25}{2}\cos(2\theta - \alpha) \geq 0
\end{aligned}$$

□

Therefore, by induction, a_i and b_i must be real number for all whole numbers i. □

Because the fact that $-1 \leq b_n \leq 1$ is proven, let $b_n = \sin\theta_n$.

$$\begin{aligned}
\sin\theta_{n+1} &= \frac{4}{5}\sin\theta_n + \frac{3}{5}|\cos\theta_n| \\
&= \cos\alpha\sin\theta_n + \sin\alpha|\cos\theta_n| \text{ (Where } \alpha \text{ is the smaller angle in 3-4-5 right triangle.)}
\end{aligned}$$

From the problem, it is evident that $\theta_0 = 0°$.

$$\begin{aligned}
&1.\cos\theta_n \geq 0 \implies b_{n+1} = \sin\theta_{n+1} = \sin(\alpha + \theta_n) \\
&2.\cos\theta_n < 0 \implies b_{n+1} = \sin\theta_{n+1} = \sin(\alpha - \theta_n)
\end{aligned}$$

b_i could further be calculated.

$$\begin{aligned}
&b_0 = \sin 0° && \theta_0 = 0° \\
&b_1 = \sin\alpha && \theta_1 = \alpha \\
&b_2 = \sin 2\alpha && \theta_2 = 2\alpha \\
&b_3 = \sin 3\alpha && \theta_3 = 3\alpha \Rightarrow \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha < 0 \\
&b_4 = \sin(-2\alpha) && \theta_4 = -2\alpha \\
&b_5 = \sin(-\alpha) && \theta_5 = -\alpha \\
&b_6 = \sin 0° && \theta_6 = 0° \\
&\quad\vdots && \\
&b_{10} = \sin(-2\alpha) && \theta_{10} = -2\alpha
\end{aligned}$$

Because $b_n = \frac{a_n}{2^n}$, the case where $n = 10$ could be investigated.

$$
\begin{aligned}
b_{10} &= \frac{a_{10}}{2^{10}} \\
a_{10} &= b_{10} \cdot 2^{10} \\
a_{10} &= \sin(-2\alpha) \cdot 2^{10} \\
&= -2^{10} \sin 2\alpha \\
&= -2^{10} \cdot 2 \sin\alpha \cos\alpha \\
&= -2^{10} \cdot 2 \cdot \frac{3}{5} \cdot \frac{4}{5} \\
&= \boxed{-\frac{24576}{25}}
\end{aligned}
$$

□

Problem

Show that if $x + y + z = xyz$, then $\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}$.

Proof. The form $\frac{2x}{1-x^2}$ looks familiar. Therefore, let $x = \tan\alpha$, $y = \tan\beta$, and $z = \tan\gamma$.

$$
\begin{aligned}
\tan\alpha + \tan\beta + \tan\gamma &= \tan\alpha\tan\beta\tan\gamma \\
\tan(\alpha+\beta+\gamma) &= \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - (\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha)} = 0
\end{aligned}
$$

However, notice that $\alpha + \beta + \gamma = k\pi$ for some integer k since $\tan(\alpha+\beta+\gamma) = 0$. In other words, $\tan(2\alpha + 2\beta + 2\gamma) = 0$.

$$
\begin{aligned}
\tan(2\alpha + 2\beta + 2\gamma) &= 0 \\
\frac{\tan 2\alpha + \tan 2\beta + \tan 2\gamma - \tan 2\alpha\tan 2\beta\tan 2\gamma}{1 - (\tan 2\alpha\tan 2\beta + \tan 2\beta\tan 2\gamma + \tan 2\gamma\tan 2\alpha)} &= 0 \\
\tan 2\alpha + \tan 2\beta + \tan 2\gamma - \tan 2\alpha\tan 2\beta\tan 2\gamma &= 0 \\
\tan 2\alpha + \tan 2\beta + \tan 2\gamma &= \tan 2\alpha\tan 2\beta\tan 2\gamma \\
\frac{2\tan\alpha}{1-\tan^2\alpha} + \frac{2\tan\beta}{1-\tan^2\beta} + \frac{2\tan\gamma}{1-\tan^2\gamma} &= \frac{2\tan\alpha}{1-\tan^2\alpha} \cdot \frac{2\tan\beta}{1-\tan^2\beta} \cdot \frac{2\tan\gamma}{1-\tan^2\gamma} \\
\therefore \frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} &= \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}
\end{aligned}
$$

□

2000 AIME II Problem 15

Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$

Solution

Key Word Conjecture and Proof

First, multiply $\sin n^\circ$ on both sides.

$$\begin{aligned}\frac{\sin n^\circ}{\sin m^\circ \sin(m+1)^\circ} &= \frac{\sin(k+n-k)^\circ}{\sin m^\circ \sin(m+1)^\circ} \\ &= \frac{\sin(k+n)^\circ \cos k^\circ - \sin k^\circ \cos(k+n)^\circ}{\sin m^\circ \sin(m+1)^\circ}\end{aligned}$$

Let $k = m$ since k is could be any number.

$$\begin{aligned}&\frac{\sin(k+n)^\circ \cos k^\circ}{\sin m^\circ \sin(m+1)^\circ} - \frac{\sin k^\circ \cos(k+n)^\circ}{\sin m^\circ \sin(m+1)^\circ} \\ =& \frac{\sin(m+n)^\circ \cos m^\circ}{\sin m^\circ \sin(m+1)^\circ} - \frac{\sin m^\circ \cos(m+n)^\circ}{\sin m^\circ \sin(m+1)^\circ}\end{aligned}$$

Lemma. *n is equal to 1.*

Proof.

$$\begin{aligned}&= \frac{\sin(m+1)^\circ \cos m^\circ}{\sin m^\circ \sin(m+1)^\circ} - \frac{\sin m^\circ \cos(m+1)^\circ}{\sin m^\circ \sin(m+1)^\circ} \\ &= \frac{\cos m^\circ}{\sin m^\circ} - \frac{\cos(m+1)^\circ}{\sin(m+1)^\circ} \\ &= \cot m^\circ - \cot(m+1)^\circ\end{aligned}$$

The sum of all numbers could be written. Moreover, notice that $\cot\alpha + \cot\beta = 0$ if $\alpha + \beta = 180^\circ$.

$$\begin{aligned}&\cot 45^\circ - \cot 46^\circ + \cot 47^\circ - \cot 48^\circ + \cdots - \cot 132^\circ + \cot 133^\circ - \cot 134^\circ \\ =& (\cot 45^\circ + \cot 47^\circ + \cdots + \cot 89^\circ + \cot 91^\circ + \cdots + \cot 133^\circ) \\ &\qquad - (\cot 46^\circ + \cdots + \cot 88^\circ + \cot 90^\circ + \cot 92^\circ + \cdots + \cot 134^\circ) \\ =& \cot 45^\circ - \cot 90^\circ \\ =& 1\end{aligned}$$

Because $1 = 1$, the lemma is true. □

n could be 1. Moreover, there are no smaller positive integer less than 1 to test. Thus, the least positive integer n that satisfies the given condition is $\boxed{001}$. □

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Problem

Prove that $\sum_{k=1}^{n} \arctan \frac{1}{2k^2} = \arctan \frac{n}{n+1}$.

Proof. Because inverse trigonometric function is given, let $x = \arctan \alpha$ and $y = \arctan \beta$. Therefore, $\tan x = \alpha$ and $\tan y = \beta$.

$$\begin{aligned} \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\ &= \frac{\alpha + \beta}{1 - \alpha\beta} \\ \therefore \arctan\left(\frac{\alpha+\beta}{1-\alpha\beta}\right) &= \arctan \alpha + \arctan \beta \end{aligned}$$

Because the formula for each factor is given and a formula for the sum must be found, inductive steps could be utilized.

Base Case

The formula must satisfy for $n = 1$.

$$\sum_{k=1}^{1} \arctan \frac{1}{2k^2} = \arctan \frac{1}{2} = \arctan \frac{1}{1+1}$$

The base case is satisfied.

Induction

Lemma. *If* $\sum_{k=1}^{i} \arctan \frac{1}{2k^2} = \arctan \frac{i}{i+1}$ *is true for some natural number* i*, then* $\sum_{k=1}^{i+1} \arctan \frac{1}{2k^2} = \arctan \frac{(i+1)}{(i+1)+1}$ *is also true.*

Proof.

$$\begin{aligned} \sum_{k=1}^{i+1} \arctan \frac{1}{2k^2} &= \sum_{k=1}^{i} \arctan \frac{1}{2k^2} + \arctan \frac{1}{2(i+1)^2} \\ &= \arctan \frac{i}{i+1} + \arctan \frac{1}{2(i+1)^2} \\ &= \arctan\left(\frac{\frac{i}{i+1} + \frac{1}{2(i+1)^2}}{1 - \frac{i}{i+1} \cdot \frac{1}{2(i+1)^2}}\right) = \arctan\left(\frac{\frac{2(i+1)i+1}{2(i+1)^2}}{\frac{2(i+1)^3-i}{2(i+1)^3}}\right) \\ &= \arctan\left(\frac{\frac{2(i+1)i+1}{1}}{\frac{2(i+1)^3-i}{i+1}}\right) = \arctan\left(\frac{(i+1)(2(i+1)i+1)}{2(i+1)^3-i}\right) \\ &= \arctan\left(\frac{(i+1)(2(i+1)i+1)}{2i^3+6i^2+5i+2}\right) = \arctan\left(\frac{(i+1)(2i^2+2i+1)}{2i^3+6i^2+5i+2}\right) = \arctan\left(\frac{(i+1)(2i^2+2i+1)}{(i+2)(2i^2+2i+1)}\right) \\ &= \arctan\left(\frac{i+1}{i+2}\right) \end{aligned}$$

□

By induction, the equation $\sum_{k=1}^{n} \arctan \frac{1}{2k^2} = \arctan \frac{n}{n+1}$ is true. □