2021 Fall AMC 12B Problem 22

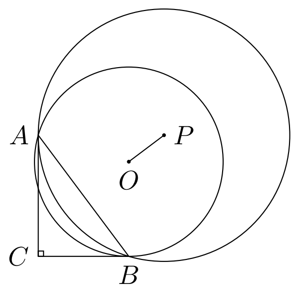
Right triangle ABC has side lengths $BC = 6$, $AC = 8$, and $AB = 10$. A circle centered at O is tangent to line BC at B and passes through A. A circle centered at P is tangent to line AC at A and passes through B. What is OP?

(A) $\frac{23}{8}$ **(B)** $\frac{29}{10}$ **(C)** $\frac{35}{12}$ **(D)** $\frac{73}{25}$ **(E)** 3

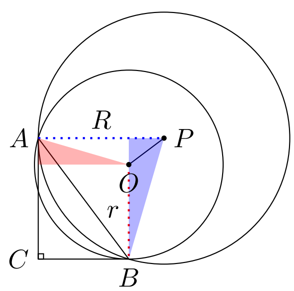
Solution

Key Word Pythagorean Theorem

First, the diagram could be drawn. Moreover, finding the distance using Analytic Geometry seems easier.



Using the fact that the circles are tangent to the triangle, the Pythagorean Theorem may be applicable. The colored triangle may be exploited.



Therefore, using the Pythagorean Theorem, $6^2 + (8 - r)^2 = r^2$ and $8^2 + (R - 6)^2 = R^2$ are true.

$$6^2 + (8 - r)^2 = r^2$$
$$16r = 100$$
$$r = \frac{25}{4}$$

$$8^2 + (R - 6)^2 = R^2$$
$$12R = 100$$
$$R = \frac{25}{3}$$

Therefore, using the center of the circles,

$$OP = \sqrt{(8 - \frac{25}{4})^2 + (\frac{25}{3} - 6)^2}$$
$$= \boxed{\textbf{(C)}\ \frac{35}{12}}.$$

□

2021 Fall AMC 12B Problem 24

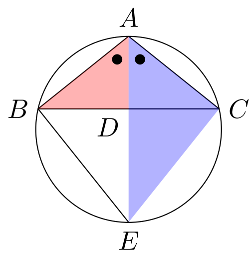
Triangle ABC has side lengths $AB = 11, BC = 24$, and $CA = 20$. The bisector of $\angle BAC$ intersects $\overline{BC}$ in point D, and intersects the circumcircle of $\triangle ABC$ in point $E \neq A$. The circumcircle of $\triangle BED$ intersects the line AB in points B and $F \neq B$. What is CF?

(A) 28 **(B)** $20\sqrt{2}$ **(C)** 30 **(D)** 32 **(E)** $20\sqrt{3}$

Solution

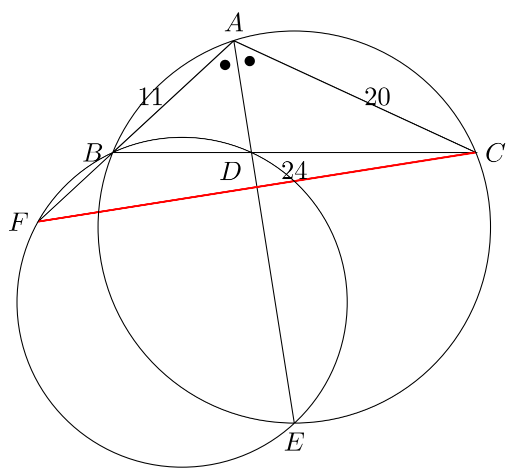
Key Word Similar Triangle, Power of a Point Theorem, Stewart's Theorem, Law of Cosines

Property: Derived from a Pair of Similar Triangle



$AB \cdot AC = AD \cdot AE$ ($\because \triangle ADB \sim \triangle ACE$)

First, the situation could be drawn.



Using the similar triangle, notice that $AD \cdot AE = 11 \cdot 20$. Moreover, using the Power of a Point Theorem, it is evident that $AF = 20$. Two possible pathways are resulted.

Stewart's Theorem

$$\begin{aligned} 11 \cdot FC^2 + 9 \cdot 20^2 &= (11+9)(11 \cdot 9 + 24^2) \\ 11FC^2 + 3600 &= (20)(99 + 576) \\ 11FC^2 + 3600 &= 20 \cdot 675 \\ 11FC^2 &= 9900 \end{aligned}$$

$\therefore FC =$ **(C)** 30

Law of Cosines

$$\begin{aligned} \frac{11^2 + 20^2 - 24^2}{2 \cdot 11 \cdot 20} &= \frac{20^2 + 20^2 - FC^2}{2 \cdot 20 \cdot 20} \\ -5 &= \frac{800 - FC^2}{20} \\ -100 &= 800 - FC^2 \end{aligned}$$

$\therefore FC =$ **(C)** 30

□

Problem

Compute the smallest positive angle x, in degrees, such that $\tan 4x = \frac{\cos x - \sin x}{\cos x + \sin x}$.

Solution

Key Word Trigonometric Identities

$$\begin{aligned}
\frac{\sin 4x}{\cos 4x} &= \frac{\cos x - \sin x}{\cos x + \sin x} \\
\sin 4x \cdot \cos x + \sin 4x \dot{\sin} x &= \cos x \cdot \cos 4x - \sin x \cdot \cos 4x \\
\sin 4x \cdot \cos x + \sin x \cdot \cos 4x &= \cos x \cdot \cos 4x - \sin 4x \dot{\sin} x \\
\sin(x + 4x) &= \cos(x + 4x) \\
\sin 5x &= \cos 5x \\
\therefore x \Rightarrow & \boxed{9^\circ}
\end{aligned}$$

□

Problem

Prove that for all real x and y, we have $-\frac{1}{2} \leq \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \leq \frac{1}{2}$.

Proof. Let $x = \tan \alpha$ and $y = \tan \beta$.

$$\begin{aligned}
\frac{(\tan \alpha + \tan \beta)(1 - \tan \alpha \cdot \tan \beta)}{(1 + \tan^2 \alpha)(1 + \tan^2 \beta)} &= \frac{(\tan \alpha + \tan \beta)(1 - \tan \alpha \cdot \tan \beta)}{\sec^2 \alpha \cdot \sec^2 \beta} \\
&= \left(\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}\right)\left(1 - \frac{\sin \alpha \cdot \sin \beta}{\cos \alpha \cdot \cos \beta}\right)(\cos^2 \alpha \cdot \cos^2 \beta) \\
&= (\sin \alpha \cos \beta + \sin \beta \cos \alpha)(\cos \alpha \cos \beta - \sin \alpha \sin \beta) \\
&= \sin(\alpha + \beta)\cos(\alpha + \beta) \\
&= \frac{1}{2}\sin(2(\alpha + \beta))
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{2} \leq & \frac{1}{2}\sin(2(\alpha + \beta)) \leq \frac{1}{2} \\
\therefore -\frac{1}{2} \leq & \frac{(x + y)(1 - xy)}{(1 + x^2)(1 + y^2)} \leq \frac{1}{2}
\end{aligned}$$

□

Problem

Determine all θ such that $0 \leq \theta \leq \frac{\pi}{2}$ and $\sin^5\theta + \cos^5\theta = 1$.

Solution

Key Word

For $|r| < 1$,

$$\lim_{n\to\infty} |r|^n = 0.$$

Moreover, $-1 \leq |\sin\theta||\cos\theta| \leq 1$. Using the first fact, the following inequalities and equations are true.

$$\cos^2\theta \geq \cos^5\theta$$
$$\sin^2\theta \geq \cos^5\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^5\theta + \cos^5\theta = 1$$

In order to satisfy the given conditions, $\sin^2\theta = \sin^5\theta$ and $\cos^2\theta = \cos^5\theta$ must be true.

$$1 = \sin^3 \qquad \text{Unless } 0$$
$$1 = \cos^3 \qquad \text{Unless } 0$$

Therefore, $\theta = \boxed{0, \frac{\pi}{2}}$ □

Problem

Suppose that $\sec\theta + \tan\theta = \frac{22}{7}$. Find $\csc\theta + \cot\theta$.

Solution

Key Word MUST THINK OF THIS WHEN YOU SEE $\sec\theta + \tan\theta$!!

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$\tan^2\theta + 1 = \sec^2\theta$$

$$2\sec\theta = \frac{533}{154}$$
$$\therefore \sec\theta = \frac{533}{308}$$

$$2\tan\theta = \frac{435}{154}$$
$$\therefore \tan\theta = \frac{435}{308}$$
$$\therefore \cot\theta = \frac{1}{\frac{435}{308}} = \frac{308}{435}$$

$$(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$$
$$(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$$
$$\sec\theta + \tan\theta = \frac{22}{7},\ \sec\theta - \tan\theta = \frac{7}{22}$$

$$\sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{435}{533}$$
$$\therefore \csc\theta = \frac{533}{435}$$

$$\therefore \csc\theta + \cot\theta = \frac{533}{435} + \frac{308}{435} = \frac{841}{435} = \boxed{\frac{29}{15}}$$

□