Problem

If $S = \frac{1}{\frac{1}{1980}+\frac{1}{1981}+\cdots+\frac{1}{1997}}$, what is the integer part of S?

Solution

Key Word Properties of Fraction

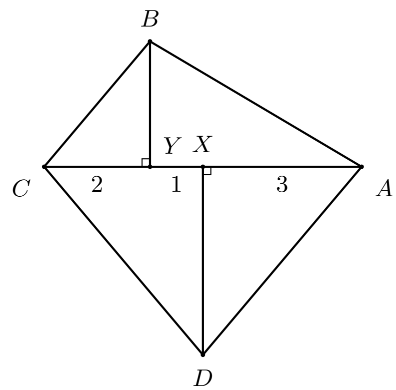
First and foremost, when the range of the denominator could be found, the range of S would also be obtained. Let $S' = \frac{1}{1980} + \frac{1}{1981} + \cdots + \frac{1}{1997}$

$$\frac{1}{1980} + \cdots + \frac{1}{1980} > S' > \frac{1}{1997} + \cdots + \frac{1}{1997}$$
$$\frac{18}{1980} > S' > \frac{18}{1997}$$
$$\frac{1}{\frac{18}{1980}} < S < \frac{1}{\frac{18}{1997}}$$
$$\frac{1980}{18} < S < \frac{1997}{18}$$
$$110 < S < 110.9444$$

Therefore, the integer part of S is $\boxed{110}$. □

2021 Fall AMC 12A Problem 21

Let $ABCD$ be an isosceles trapezoid with $\overline{BC} \parallel \overline{AD}$ and $AB = CD$. Points X and Y lie on diagonal $\overline{AC}$ with X between A and Y, as shown in the figure. Suppose $\angle AXD = \angle BYC = 90^\circ$, $AX = 3$, $XY = 1$, and $YC = 2$. What is the area of $ABCD$?

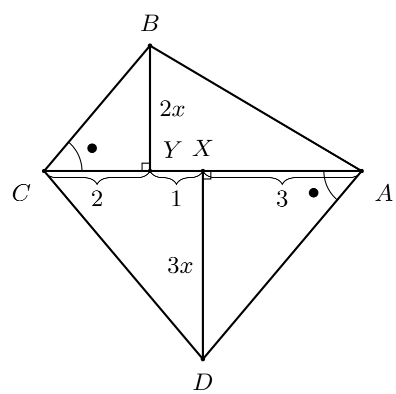


(A) 15 **(B)** $5\sqrt{11}$ **(C)** $3\sqrt{35}$ **(D)** 18 **(E)** $7\sqrt{7}$

Solution

Key Word Pythagorean Theorem

First, different ratios and relationships could be found through the diagram.



Using the Pythagorean Theorem, the following equations are true.

$$(2x)^2 + 4^2 = 3^2 + (3x)^2$$
$$4x^2 + 16 = 9 + 9x^2$$
$$5x^2 = 7$$
$$x^2 = \frac{7}{5}$$

$$\frac{1}{2} \cdot 6 \cdot 5x = 15 \cdot \sqrt{\frac{7}{5}}$$
$$= \boxed{\textbf{(C)}\ 3\sqrt{35}}$$

□

2021 Fall AMC 12A Problem 23

A quadratic polynomial with real coefficients and leading coefficient 1 is called *disrespectful* if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

(A) $\frac{5}{16}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{5}{8}$ **(D)** 1 **(E)** $\frac{9}{8}$

Solution

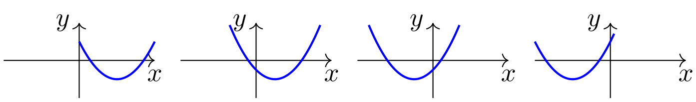
Key Word Graphing Quadratic Functions

Because $p(x)$ is a quadratic equation with a leading coefficient of 1, let $p(x) = x^2 + ax + b$. Moreover, let r_1, r_2, r_3 be the roots of the equation $p(p(x)) = 0$.

$$p(r_1) = a$$
$$p(r_2) = b$$
$$p(r_3) = c$$

$$p(a) = p(b) = p(c) = 0$$
$$p(x) = (x - \alpha)(x - \beta)$$

The following cases are the possible graphs for $p(x)$.



The second graph is the case that could provide the maximum sum of the roots.

$$\therefore \left(x - \frac{\alpha+\beta}{2}\right)^2 + \alpha = (x-\alpha)(x-\beta)$$
$$x^2 - (\alpha+\beta)x + \frac{(\alpha+\beta)^2}{4} + \alpha = x^2 - (\alpha+\beta)x + \alpha\beta$$
$$\frac{(\alpha+\beta)^2}{4} + \alpha = \alpha\beta$$
$$(\alpha+\beta)^2 = 4\alpha\beta - 4\alpha = 4\alpha(\beta - 1)$$

$$\sqrt{\alpha(\beta-1)} \leq \frac{\alpha + (\beta-1)}{2}$$
$$\textit{leads to}$$
$$\alpha = \beta - 1.$$

$$(\alpha+\beta)^2 = 4\alpha^2$$
$$\alpha = \beta \textit{ or } 3\alpha + \beta = 0$$

$$\therefore \alpha = -\frac{1}{4}, \beta = \frac{3}{4}$$

$$p(1) = (1-\alpha)(1-\beta) = \frac{5}{4} \cdot \frac{1}{4} = \boxed{\textbf{(A)}\ \frac{5}{16}}$$

□

Uploaded a new solution in AOPS!!