

Basic Trigonometric Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} \Rightarrow 1 + \cot^2 \theta = \csc^2 \theta$$

$$\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} \Rightarrow \tan^2 \theta + 1 = \sec^2 \theta$$

$$\begin{cases} \sin(2\pi + \theta) = \sin \theta \\ \cos(2\pi + \theta) = \cos \theta \\ \tan(\pi + \theta) = \tan \theta \end{cases} \quad \begin{cases} \sin(-\theta) = -\sin \theta \\ \cos(-\theta) = \cos \theta \\ \tan(-\theta) = -\tan \theta \end{cases}$$

$$\begin{cases} \sin(\pi - \theta) = \sin \theta \\ \cos(\pi - \theta) = -\cos \theta \\ \tan(\pi - \theta) = -\tan \theta \end{cases} \quad \begin{cases} \sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \\ \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \\ \tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan \theta} \end{cases}$$

Example.

Calculate $(1 - \sin^2 \theta)(1 + \tan^2 \theta)$.

Solution

$$\begin{aligned} (1 - \sin^2 \theta)(1 + \tan^2 \theta) &= 1 + \tan^2 \theta - \sin^2 \theta - \sin^2 \theta \tan^2 \theta \\ &= \sec^2 \theta - \sin^2 \theta(1 + \tan^2 \theta) \\ &= \sec^2 \theta - \sin^2 \theta \sec^2 \theta \\ &= \sec^2 \theta(1 - \sin^2 \theta) \\ &= \sec^2 \theta \cdot \cos^2 \theta \\ &= \boxed{1} \end{aligned}$$

□

Example. Prove that

$$\frac{1 + \tan \theta}{1 - \tan \theta} = \frac{1 + 2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}.$$

Proof.

$$\begin{aligned} \frac{1 + \tan \theta}{1 - \tan \theta} &= \frac{1 + \frac{\sin \theta}{\cos \theta}}{1 - \frac{\sin \theta}{\cos \theta}} \\ &= \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \\ &= \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \cdot \frac{\cos \theta + \sin \theta}{\cos \theta + \sin \theta} \\ &= \frac{\cos^2 \theta + 2 \sin \theta \cos \theta + \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta} \\ &= \boxed{\frac{1 + 2 \sin \theta \cos \theta}{\cos^2 \theta - \sin^2 \theta}} \end{aligned}$$

□

Example. Let $\sin \theta + \cos \theta = \frac{1}{2}$. Compute

$$\sin^6 \theta + \cos^6 \theta.$$

Solution

$$\begin{aligned} (\sin \theta + \cos \theta)^2 &= \frac{1}{4} \\ \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta &= \frac{1}{4} \\ 2 \sin \theta \cos \theta &= -\frac{3}{4} \\ \sin \theta \cos \theta &= -\frac{3}{8} \end{aligned}$$

$$\begin{aligned} \sin^6 \theta + \cos^6 \theta &= (\sin^2 \theta)^3 + (\cos^2 \theta)^3 \\ &= (\sin^2 \theta + \cos^2 \theta)^3 - 3 \sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta) \\ &= 1 - 3 \sin^2 \theta \cos^2 \theta \\ &= 1 - 3 \left(-\frac{3}{8} \right)^2 \\ &= 1 - 3 \cdot \frac{9}{64} \\ &= \boxed{\frac{37}{64}} \end{aligned}$$

□

Example. Let $\sin \theta + \cos \theta = \frac{1}{3}$. Compute

$$\frac{1}{\cos \theta} \left(\tan \theta + \frac{1}{\tan^2 \theta} \right).$$

Solution

$$(\sin \theta + \cos \theta)^2 = \frac{1}{9}$$

$$1 + 2 \sin \theta \cos \theta = \frac{1}{9}$$

$$\sin \theta \cos \theta = -\frac{4}{9}$$

$$\begin{aligned} \frac{1}{\cos \theta} \left(\tan \theta + \frac{1}{\tan^2 \theta} \right) &= \frac{1}{\cos \theta} \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} \right) \\ &= \frac{\sin \theta}{\cos^2 \theta} + \frac{\cos \theta}{\sin^2 \theta} \\ &= \frac{\sin^3 \theta + \cos^3 \theta}{\sin^2 \theta \cos^2 \theta} \\ &= \frac{\left(\frac{1}{3}\right) \left(1 + \frac{4}{9}\right)}{\left(-\frac{4}{9}\right)^2} \\ &= \frac{\frac{13}{27}}{\frac{16}{81}} \\ &= \boxed{\frac{39}{16}} \end{aligned}$$

□

Example. If the roots of the equation: $4x^2 - 4px + p^2 - 2$ are $\sin \theta$ and $\cos \theta$, then find the value of p .

Solution

$$\sin \theta + \cos \theta = p$$

$$\sin \theta \cos \theta = \frac{p^2 - 2}{4}$$

$$p^2 = 1 + 2 \left(\frac{p^2 - 2}{4} \right)$$

$$p^2 = 0$$

$$\therefore p = \boxed{0}$$

□

Example. Find the parent equation whose roots are $\tan \theta$ and $\frac{1}{\tan \theta}$ given that $\sin \theta \cos \theta = -\frac{1}{2}$.

Solution

$$x^2 - (a + b)x + ab = 0$$

$$\begin{aligned} a + b &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\ &= \frac{1}{\sin \theta \cos \theta} \\ &= -2 \end{aligned}$$

$$ab = 1$$

$$\therefore \boxed{x^2 + 2x + 1 = 0}$$

□

Example. If $\sin \theta + \cos \theta = \frac{\sqrt{2}}{2}$, find the parent equation whose roots are $\sin^3 \theta$ and $\cos^3 \theta$.

Solution

$$\begin{aligned} (\sin \theta + \cos \theta)^2 &= \frac{1}{2} \\ \sin \theta \cos \theta &= -\frac{1}{4} \end{aligned}$$

$$\begin{aligned} \sin^3 \theta + \cos^3 \theta &= \left(\frac{\sqrt{2}}{2}\right) \left(1 + \frac{1}{4}\right) \\ &= \frac{5\sqrt{2}}{8} \end{aligned}$$

$$\sin^3 \theta \cos^3 \theta = -\frac{1}{64}$$

$$\boxed{x^2 - \frac{5\sqrt{2}}{8}x - \frac{1}{64} = 0}$$

□

Example. Find the value of $\tan \theta$ if $1 + \sin^2 \theta = 3 \sin \theta \cos \theta$.

Solution

$$\begin{aligned} 1 + \sin^2 \theta &= 3 \sin \theta \cos \theta \\ 2 \sin^2 \theta + \cos^2 \theta &= 3 \sin \theta \cos \theta \\ 2 \tan^2 \theta + 1 &= 3 \tan \theta \\ (2 \tan \theta - 1)(\tan \theta - 1) &= 0 \\ \therefore \tan \theta &= \boxed{\frac{1}{2} \text{ or } 1} \end{aligned}$$

□

Example. Compute

$$\cos \frac{59}{6} \pi \cdot \tan \frac{37}{6} \pi + \sin \left(-\frac{26}{3} \pi \right) \cdot \tan \frac{11}{4} \pi.$$

Solution

$$\begin{aligned} &\cos \frac{59}{6} \pi \cdot \tan \frac{37}{6} \pi + \sin \left(-\frac{26}{3} \pi \right) \cdot \tan \frac{11}{4} \pi \\ &= \cos \frac{1}{6} \pi \cdot \tan \frac{1}{6} \pi - \sin \left(\frac{2}{3} \pi \right) \cdot \tan \frac{3}{4} \pi \\ &= \cos 30^\circ \cdot \tan 30^\circ + \cos 30^\circ \cdot \frac{1}{\tan 45^\circ} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2} \\ &= \boxed{\frac{1 + \sqrt{3}}{2}} \end{aligned}$$

□

Example. Find the value of $\cos \left(n\pi + \frac{(-1)^n \pi}{3} \right)$ if n is an integer.

Solution

n is Even

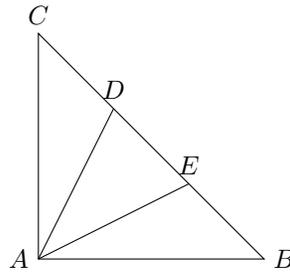
$$\begin{aligned} \cos \left(n\pi + \frac{(-1)^n \pi}{3} \right) &= \cos \left(2k\pi + \frac{(-1)^{2k} \pi}{3} \right) \\ &= \cos \left(\frac{\pi}{3} \right) \\ &= \frac{1}{2} \end{aligned}$$

n is Odd

$$\begin{aligned} \cos \left(n\pi + \frac{(-1)^n \pi}{3} \right) &= \cos \left((2k+1)\pi + \frac{(-1)^{2k+1} \pi}{3} \right) \\ &= \cos \left(\pi - \frac{\pi}{3} \right) \\ &= -\frac{1}{2} \\ \therefore \cos \left(n\pi + \frac{(-1)^n \pi}{3} \right) &= \pm \frac{1}{2} \end{aligned}$$

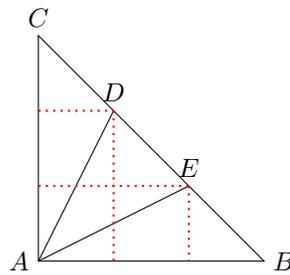
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Example. For a right triangle $\triangle ABC$, let $AD = DE = EB$, $AD = \cos \theta + \sin \theta$, and $AE = \cos \theta - \sin \theta$. Find the length of BC .



Solution

First, using the properties of congruent triangles, draw the dotted lines.



Let $AB = 3a$ and $AC = 3b$. Using Pythagorean Theorem, it is evident that $4a^2 + b^2 = 1 - 2 \sin \theta \cos \theta$ and $a^2 + 4b^2 = 1 + 2 \sin \theta \cos \theta$. In other words, $a^2 + b^2 = \frac{2}{5}$.

$$\therefore BC = 9 \cdot \frac{2}{5} = \boxed{\frac{18}{5}}$$

□

Example. If the point $(a + \cos \theta, \frac{a}{2} + \sin \theta)$ is in the circle $x^2 + y^2 = 4$, find the range of value a .

Solution

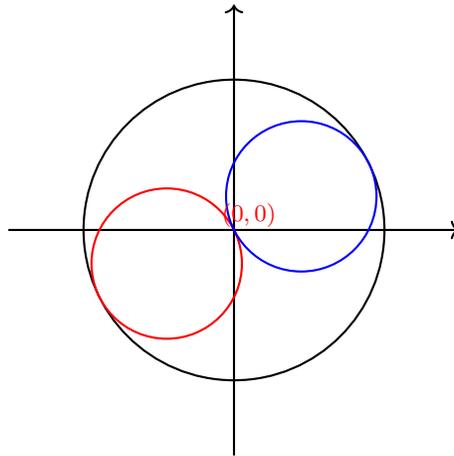
Let the point $(a + \cos \theta, \frac{a}{2} + \sin \theta)$ be (x, y) .

$$x - a = \cos \theta$$

$$y - \frac{a}{2} = \sin \theta$$

$$\therefore (x - a)^2 + \left(y - \frac{a}{2}\right)^2 = 1$$

The situation could be graphed.



Using the distance formula,

$$0 \leq \sqrt{a^2 + \left(\frac{a}{2}\right)^2} \leq 1$$

$$0 \leq a^2 + \left(\frac{a}{2}\right)^2 \leq 1$$

$$0 \leq \frac{5a^2}{4} \leq 1$$

$$0 \leq a^2 \leq \frac{4}{5}$$

$$\boxed{-\frac{2\sqrt{5}}{5} \leq a \leq \frac{2\sqrt{5}}{5}}$$

□