2021 Fall AMC 12A Problem 16

An organization has 30 employees, 20 of whom have a brand A computer while the other 10 have a brand B computer. For security, the computers can only be connected to each other and only by cables. The cables can only connect a brand A computer to a brand B computer. Employees can communicate with each other if their computers are directly connected by a cable or by relaying messages through a series of connected computers. Initially, no computer is connected to any other. A technician arbitrarily selects one computer of each brand and installs a cable between them, provided there is not already a cable between that pair. The technician stops once every employee can communicate with each other. What is the maximum possible number of cables used?

(A) 190 **(B)** 191 **(C)** 192 **(D)** 195 **(E)** 196

Solution

Key Word Worst Case Scenario

The worst case scenario would be achieved when all except one computer are connected. Therefore, by isolating a brand A computer, $19 \cdot 10 + 1 = \boxed{\textbf{(B)}\ 191}$. □

2021 Fall AMC 12A Problem 18

Each of 20 balls is tossed independently and at random into one of 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

(A) 1 **(B)** 4 **(C)** 8 **(D)** 12 **(E)** 16

Solution

Key Word Counting Strategy

With alignments of balls and bins, the following equations could be obtained.

$$p = \frac{({}_{20}C_3 + {}_{17}C_5 + {}_{12}C_4 + {}_{8}C_4 + {}_{4}C_4) \cdot \frac{5!}{3!}}{5^{20}}$$
$$q = \frac{({}_{20}C_4 + {}_{16}C_4 + {}_{12}C_4 + {}_{8}C_4 + {}_{4}C_4) \cdot \frac{5!}{5!}}{5^{20}}$$

$$\therefore \frac{p}{q} = \frac{\frac{({}_{20}C_3+{}_{17}C_5+{}_{12}C_4+{}_{8}C_4+{}_{4}C_4) \cdot \frac{5!}{3!}}{5^{20}}}{\frac{({}_{20}C_4+{}_{16}C_4+{}_{12}C_4+{}_{8}C_4+{}_{4}C_4) \cdot \frac{5!}{5!}}{5^{20}}} = \frac{({}_{20}C_3 + {}_{17}C_5 + {}_{12}C_4 + {}_{8}C_4 + {}_{4}C_4) \cdot \frac{5!}{3!}}{({}_{20}C_4 + {}_{16}C_4 + {}_{12}C_4 + {}_{8}C_4 + {}_{4}C_4)}$$
$$= \frac{\frac{20!}{3!5!4!4!4!} \cdot \frac{5!}{3!}}{\frac{20!}{4!4!4!4!4!}} = \frac{4!4!4!4!4!5!}{3!5!4!4!4!3!} = \frac{4!4!}{3!3!}$$
$$= \boxed{\textbf{(E)}\ 16}$$

□

2021 Fall AMC 12B Problem 13

Let $c = \frac{2\pi}{11}$. What is the value of

$$\frac{\sin 3c \cdot \sin 6c \cdot \sin 9c \cdot \sin 12c \cdot \sin 15c}{\sin c \cdot \sin 2c \cdot \sin 3c \cdot \sin 4c \cdot \sin 5c}?$$

(A) -1 **(B)** $-\frac{\sqrt{11}}{5}$ **(C)** $\frac{\sqrt{11}}{5}$ **(D)** $\frac{10}{11}$ **(E)** 1

Solution

Key Word Properties of Trigonometric Functions

First and foremost, $\sin\theta = \sin(\pi - \theta) = \sin(\theta + 2\pi)$ is proven to be true. Moreover, knowing the convention of AMC, c could be substituted first.

$$\begin{aligned}
\frac{\sin 3c \cdot \sin 6c \cdot \sin 9c \cdot \sin 12c \cdot \sin 15c}{\sin c \cdot \sin 2c \cdot \sin 3c \cdot \sin 4c \cdot \sin 5c} &= \frac{\sin\frac{6\pi}{11} \cdot \sin\frac{12\pi}{11} \cdot \sin\frac{18\pi}{11} \cdot \sin\frac{24\pi}{11} \cdot \sin\frac{30\pi}{11}}{\sin\frac{2\pi}{11} \cdot \sin\frac{4\pi}{11} \cdot \sin\frac{6\pi}{11} \cdot \sin\frac{8\pi}{11} \cdot \sin\frac{10\pi}{11}} \\
&= \frac{\sin\frac{6\pi}{11}}{\sin\frac{6\pi}{11}} \cdot \frac{\sin\frac{12\pi}{11}}{\sin\frac{10\pi}{11}} \cdot \frac{\sin\frac{18\pi}{11}}{\sin\frac{4\pi}{11}} \cdot \frac{\sin\frac{24\pi}{11}}{\sin\frac{2\pi}{11}} \cdot \frac{\sin\frac{30\pi}{11}}{\sin\frac{8\pi}{11}} \\
&= \frac{1}{1} \cdot \frac{-1}{1} \cdot \frac{-1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \\
&= \boxed{\textbf{(E)}\ 1}
\end{aligned}$$

□