Problem

A polynomial $f(x)$ leaves a remainder of $x^2 + x + 1$ when divided by $(x-1)^3$, and a remainder of $3x + 2$ when divided by $(x-2)^2$. What is the remainder when $f(x)$ is divided by $(x-1)^2(x-2)$?

Solution

Key Word Irreducible Formula

First of all, let's try writing what we know.

$$\begin{aligned} f(x) &= (x-1)^3Q(x) + (x^2+x+1) && \Rightarrow \quad f(1) = 3 \\ &= (x-2)^2Q'(x) + (3x+2) && \Rightarrow \quad f(2) = 8 \\ &= (x-1)^2(x-2)Q''(x) + (ax^2+bx+c) \end{aligned}$$

What can we do? First, we know that we have $(x-1)^2$ in our third form and $(x-1)^3$ in our first one. Hmm... Can we utilize them? Let's manipulate our first form first.

$$f(x) = (x-1)^2\big[(x-1)Q(x) + 1\big] + 3x$$

Knowing this, can we also change the form of our third equation?

$$\begin{aligned} f(x) &= (x-1)^2\big[(x-2)Q''(x) + \underset{\sim\sim\sim}{\quad}\big] + 3x \\ &= (x-1)^2\big[(x-2)Q''(x) + a\big] + 3x \end{aligned}$$

In other words, $ax^2 + bx + c = a(x-1)^2 + 3x$. Because we know that $f(2) = 8$, $a = 2$. Yay! We got our remainder!! Substitute 2 for a, and the remainder comes out to be $\boxed{2x^2 - x + 2}$. □

Problem

Factor $2x^4 - 3x^3y + x^2y^2 - 8xy^3 + 4y^4$.

Solution

Key Word Treating a Variable as 1

GENERAL RULE OF THUMB: If you need to factor a complex polynomial with two variable, treat one variable as one, and substitute that again.

Let $y = 1$.

$$\begin{aligned} 2x^4 - 3x^3 + x^2 - 8x + 4 &= (x-2)(2x^3 + x^2 + 3x - 2) \\ &= (x-2)(2x-1)(x^2+x+2) \\ &\Rightarrow \boxed{(x-2y)(2x-y)(x^2+xy+2y^2)} \end{aligned}$$

□

Problem

For the polynomial $P(x) = x^5 + x^4 + x^3 + x^2 + x + 1$, find the remainder when $P(x^6)$ is divided by $P(x)$.

Solution

Key Word $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1$

First, we know that $x^6 - 1 = (x-1)(x^5 + x^4 + x^3 + x^2 + x + 1)$. We also know that 30, 24, 18, 12 and 6 are multiples of 6. Can we use that?

$$\begin{aligned} P(x^6) = (x^{30} - 1) & \qquad (x^6-1)(x^{24}+x^{18}+x^{12}+x^6+1) \\ + (x^{24} - 1) & \qquad (x^6-1)(x^{18}+x^{12}+x^6+1) \\ + (x^{18} - 1) & \qquad (x^6-1)(x^{12}+x^6+1) \\ + (x^{12} - 1) & \qquad (x^6-1)(x^6+1) \\ + (x^6 - 1) & \qquad (x^6-1)(1) \\ + 6 & \end{aligned}$$

Therefore, the remainder is $\boxed{6}$. □

2021 Fall AMC 12A Problem 7

A school has 100 students and 5 teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are $50, 20, 20, 5$, and 5. Let t be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let s be the average value obtained if a student was picked at random and the number of students in their class, including the student, is noted. What is $t - s$?

(A) -18.5 **(B)** -13.5 **(C)** 0 **(D)** 13.5 **(E)** 18.5

Solution

Key Word Expected Value

$$\begin{aligned} t &= 50 \cdot \frac{1}{5} + 20 \cdot \frac{1}{5} + 20 \cdot \frac{1}{5} + 5 \cdot \frac{1}{5} + 5 \cdot \frac{1}{5} \\ &= 10 + 4 + 4 + 1 + 1 \\ &= 20 \\ s &= 50 \cdot \frac{50}{100} + 20 \cdot \frac{20}{100} + 20 \cdot \frac{20}{100} + 5 \cdot \frac{5}{100} + 5 \cdot \frac{5}{100} \\ &= 25 + 4 + 4 + 0.25 + 0.25 \\ &= 33.5 \end{aligned}$$

$$\therefore t - s = 20 - 33.5 = \boxed{\textbf{(B)}\ -13.5}$$

□

2021 Fall AMC 12B Problem 21

For real numbers x, let

$$P(x) = 1 + \cos(x) + i\sin(x) - \cos(2x) - i\sin(2x) + \cos(3x) + i\sin(3x)$$

where $i = \sqrt{-1}$. For how many values of x with $0 \leq x < 2\pi$ does

$$P(x) = 0?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution

Key Word Trigonometric Identities

Euler's Formula might help.. Maybe trigonometric identities?

$$P(x) = 1 + \cos x - \cos 2x + \cos 3x + i(\sin x - \sin 2x + \sin 3x)$$

In other words, $1 + \cos x - \cos 2x + \cos 3x = 0$ and $\sin x - \sin 2x + \sin 3x = 0$. Using trigonometric identities, we know that

$$\begin{aligned}
1 + \cos x - \cos 2x + \cos 3x = 0& \\
&= 1 + \cos x - (2\cos^2 x - 1) + (4\cos^3 x - 3\cos x) \\
&= 4\cos^3 x - 2\cos^2 x - 2\cos x + 2 \\
&= 2\cos^3 x - \cos^2 x - \cos x + 1 \\
\sin x - \sin 2x + \sin 3x = 0& \\
&= \sin x - 2\sin x\cos x + 3\sin x - 4\sin^3 x \\
&= 4\sin^3 x - 4\sin x + 2\sin x\cos x \\
&= \sin x(2\sin^2 x - 2 + \cos x) \\
&= \sin x(-2\cos^2 x + \cos x).
\end{aligned}$$

$1 + \cos x - \cos 2x + \cos 3x = 0$ must satisfy when $\sin x = 0$, $\cos x = 0$ or $\cos x = \frac{1}{2}$. The cases when $\cos x = 0$ and $\cos x = \frac{1}{2}$ does not work. Moreover, for the case $\sin x = 0$ to work, $x = 0, \pi, 2\pi$. However, none of the cases work. Therefore, $\boxed{\textbf{(A)}\ 0}$ is the answer. □

Uploaded a new solution in AOPS!!