Irreducible Polynomials

$$F(x) = f(x) \cdot Q(x) + R(x)$$

The Degree of $R(x)$ < The Degree of $f(x)$ ALWAYS!!

$$\text{Often manipulated as } \frac{F(x) - R(x)}{f(x)} = Q(x).$$

Problem

Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that $P(1) = P(3) = P(5) = P(7) = a$, and $P(2) = P(4) = P(6) = P(8) = -a$. What is the smallest possible value of a?

Solution

Key Word Irreducible Polynomials

Hmm.. What can we do?
First, let's try writing the equations.

$$P(x) = (x-1)(x-3)(x-5)(x-7)Q(x) + a$$
$$P(x) = (x-2)(x-4)(x-6)(x-8)Q'(x) - a$$

Now, why don't we try substituting diverse x values in an equation?

$$P(1) = (-1)(-3)(-5)(-7)Q'(1) - a = a$$
$$P(3) = (1)(-1)(-3)(-5)Q'(3) - a = a$$
$$P(5) = (3)(1)(-1)(-3)Q'(5) - a = a$$
$$P(7) = (5)(3)(1)(-1)Q'(7) - a = a$$

$$\therefore 105Q'(1) = -15Q'(3) = 9Q'(5) = -15Q'(7) = 2a$$

Notice that a must be a positive integer because $P(x)$ consists of integer coefficients!! In another words, the possible values for $2a$ are the multiples of $LCM(105, 15, 9)$, which is 315. However, because a must be an integer, the smallest possible value of a is $\boxed{315}$. □

Problem

Let $P(x)$ be a polynomial with integer coefficients such that $P(1) = P(3) = P(5) = P(7) = 315$, and $P(2) = P(4) = P(6) = P(8) = -315$. What is $P(x)$ with the smallest leading degree?

Solution

Key Word Irreducible Polynomials

First, let's try writing an equation for $P(x)$.

$$P(x) = (x-2)(x-4)(x-6)(x-8)Q(x) - 315$$

Next, why don't we substitute known values?

$$\begin{aligned} P(1) &= 105Q(1) = 630 \\ P(3) &= -15Q(3) = 630 \\ P(5) &= 9Q(5) = 630 \\ P(7) &= -15Q(7) = 630 \end{aligned}$$

Here, we know that $Q(3) = Q(7) = -42$. Therefore, the following equation could be written.

$$Q(x) = (x-3)(x-7)Q'(x) - 42$$

By utilizing $P(1) = 630$ and $P(5) = 630$, we could further gain information.

$$\begin{aligned} 105(12Q'(1) - 42) &= 630 \\ 9(-4Q'(5) - 42) &= 630 \end{aligned}$$

$$\begin{aligned} Q'(1) &= 4 \\ Q'(5) &= -28 \end{aligned}$$

In order for $P(x)$ to have least degree, $Q'(x)$ must have least degree. In other words, $Q'(x) = -8x + 12$.

$$\therefore \boxed{P(x) = (x-2)(x-4)(x-6)(x-8)\{(x-3)(x-7)(-8x+12) - 42\} - 315}$$

□

Problem

For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

Solution

Key Word Vieta's Formula

The problem told us that the following expression is true.

$$f(x) = g(x) \cdot (x - r_4)$$

Moreover, using Vieta's Formula, we can get our information!

$$\begin{aligned} r_1 + r_2 + r_3 &= -a \\ r_1r_2 + r_1r_3 + r_2r_3 &= 1 \\ r_1r_2r_3 &= -10 \end{aligned}$$

$$\begin{aligned} r_1 + r_2 + r_3 + r_4 &= -1 \\ r_1r_2 + \cdots + r_3r_4 &= b \\ r_1r_2r_3 + \cdots + r_2r_3r_4 &= -100 \\ r_1r_2r_3r_4 &= c \end{aligned}$$

Using the information above, let's try writing b, c, and r_4 in different ways.

$$\begin{aligned} b &= 1 - ar_4 \\ c &= -10r_4 \\ r_4 &= a - 1 \\ r_4 &= -90 \ (\because r_1r_2r_3 + r_4(r_1r_2 + r_1r_3 + r_2r_3) = -10 + r_4 = -100 \end{aligned}$$

$$\therefore a = -89, b = -8009, c = 900$$

$$f(1) = 1 + 1 + b + 100 + c = \boxed{-7007}$$

□