2021 AMC 12B Problem 14

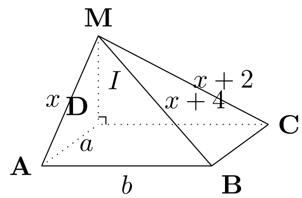
Let $ABCD$ be a rectangle and let $\overline{DM}$ be a segment perpendicular to the plane of $ABCD$. Suppose that $\overline{DM}$ has integer length, and the lengths of $\overline{MA}, \overline{MC}$, and $\overline{MB}$ are consecutive odd positive integers (in this order). What is the volume of pyramid $MABCD$?

(A) $24\sqrt{5}$ **(B)** 60 **(C)** $28\sqrt{5}$ **(D)** 66 **(E)** $8\sqrt{70}$

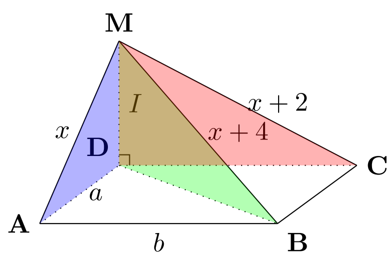
Solution

Key Word Pythagorean Theorem

First and foremost, the diagram must be drawn.



With the diagram, an impulse to utilize Pythagorean theorem is created. Thereby, the shaded right triangles may be used.



The following equations could be obtained from Pythagorean Theorem.

$$I^2 + a^2 = x^2$$
$$I^2 + b^2 = (x+2)^2$$
$$I^2 + a^2 + b^2 = (x+4)^2$$

Because x and I are integers, writing an equation in terms of only x and I may be beneficial for trial and error.

$$x^2 + (x+2)^2 = (x+4)^2 + I^2$$
$$2x^2 + 4x + 4 = x^2 + 8x + 16 + I^2$$
$$x^2 - 4x - 12 = I^2$$
$$(x-6)(x+2) = I^2$$

$x = 7$ seems valid since $I = 3$ when $x = 7$. Therefore, when $x = 7$, $a = 2\sqrt{10}$ and $b = 6\sqrt{2}$. We could double check that $40 + 72 + 9 = 11^2$. Thereby,

$$\frac{a \cdot b \cdot I}{3} = \frac{2\sqrt{10} \cdot 6\sqrt{2} \cdot 3}{3} = \boxed{\textbf{(A) } 24\sqrt{5}}.$$

$\square$

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2021 AMC 12B Problem 21

Problem 21 Let S be the sum of all positive real numbers x for which

$$x^{2^{\sqrt{2}}} = \sqrt{2}^{2^x}.$$

Which of the following statements is true?

(A) $S < \sqrt{2}$ **(B)** $S = \sqrt{2}$ **(C)** $\sqrt{2} < S < 2$ **(D)** $2 \le S < 6$ **(E)** $S \ge 6$

Solution

Key Word Approximation

While utilizing log may seem conventional, graphing may also be used. Notice that $x^{2^{\sqrt{2}}}$ is a U-shaped, differentiable curve. Moreover, $\sqrt{2}^{2^x}$ is an exponentially increasing function. Furthermore, we could notice that the two graphs meet at $x = \sqrt{2}$.

x	$x^{2^{\sqrt{2}}}$	$\sqrt{2}^{2^x}$	Comparison
0	0	$\sqrt{2}$	$x^{2^{\sqrt{2}}} < \sqrt{2}^{2^x}$
1	$2 < 2^{\sqrt{2}} < 4$	2	$x^{2^{\sqrt{2}}} < \sqrt{2}^{2^x}$
$\sqrt{2}$	$\sqrt{2}^{2^{\sqrt{2}}}$	$\sqrt{2}^{2^{\sqrt{2}}}$	$x^{2^{\sqrt{2}}} = \sqrt{2}^{2^x}$
2	$4 < 2^{2^{\sqrt{2}}} < 16$	4	$x^{2^{\sqrt{2}}} > \sqrt{2}^{2^x}$
3	$9 < 3^{2^{\sqrt{2}}} < 81$	16	NA
4	$16 < 4^{2^{\sqrt{2}}} < 256$	256	$x^{2^{\sqrt{2}}} < \sqrt{2}^{2^x}$

Because the $x^2 < x^{2^{\sqrt{2}}} < x^4$ and $\sqrt{2}^{2^x}$ is an exponentially increasing function, $x^{2^{\sqrt{2}}}$ can never meet or catch up $\sqrt{2}^{2^x}$ from $x \ge 4$. In another words, only two intersections, or the roots of the equation, exists. The roots are $\sqrt{2}$ and a constant in the interval $2 < r_2 < 4$. Let $S = \sqrt{2} + r_2$. It is evident that the possible range for S is $\boxed{\textbf{(D)}\ 2 \le S < 6}$. □

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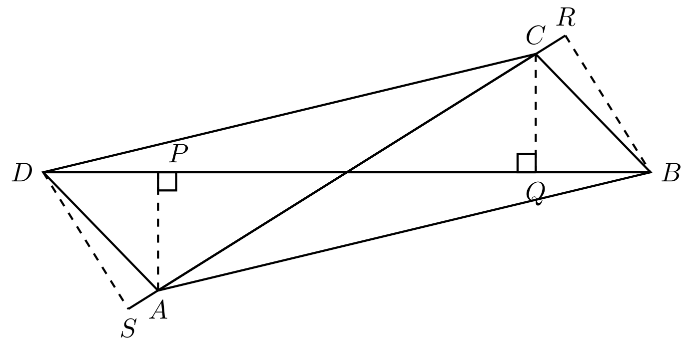
2021 AMC 12B Problem 24

Let $ABCD$ be a parallelogram with area 15. Points P and Q are the projections of A and C, respectively, onto the line BD; and points R and S are the projections of B and D, respectively, onto the line AC. See the figure, which also shows the relative locations of these points.



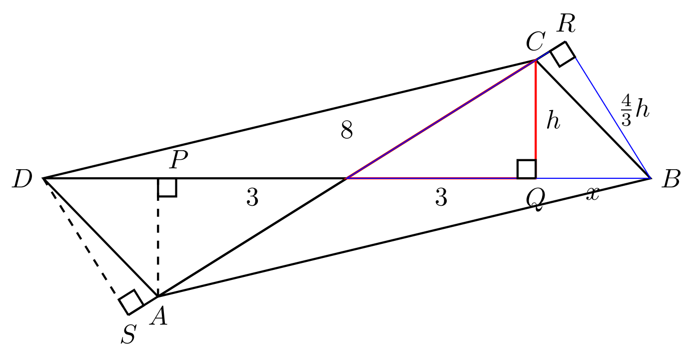
Suppose $PQ = 6$ and $RS = 8$, and let d denote the length of $\overline{BD}$, the longer diagonal of $ABCD$. Then d^2 can be written in the form $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

(A) 81 **(B)** 89 **(C)** 97 **(D)** 105 **(E)** 113

Solution

Key Word Pythagorean Theorem

First and foremost, the condition given in the problem could be represented. Moreover, let $QB = x$ and $CQ = h$ for convenience.



Using the property of a parallelogram, it could be inferred that $\frac{h(3+x)}{2} = \frac{15}{4}$. Moreover, Pythagorean Theorem manifests that $\left(\frac{4}{3}h\right)^2 + 4^2 = (3 + x)^2$.

$$3 + x = \frac{15}{2h}$$

$$\left(\frac{4}{3}h\right)^2 + 4^2 = \left(\frac{15}{2h}\right)^2$$

$$\frac{16h^2}{9} + 16 = \frac{225}{4h^2}$$

$$64h^4 + 576h^2 - 2025 = 0$$

$$h^2 \Rightarrow \frac{9\sqrt{41} - 36}{8}$$

$$\left(\frac{4}{3}h\right)^2 + 4^2 = (3+x)^2$$
$$\frac{16}{9} \cdot \frac{9\sqrt{41}-36}{8} + 4^2 = (3+x)^2$$
$$2\sqrt{41} - 8 + 16 = (3+x)^2$$
$$(3+x)^2 = 8 + 2\sqrt{41}$$
$$d^2 = 4(3+x)^2 = 32 + 8\sqrt{41}$$

Thus, $32 + 8 + 41 = \boxed{\textbf{(A) } 81}$. □

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2021 AMC 12B Problem 19

Two fair dice, each with at least 6 faces are rolled. On each face of each die is printed a distinct integer from 1 to the number of faces on that die, inclusive. The probability of rolling a sum of 7 is $\frac{3}{4}$ of the probability of rolling a sum of 10, and the probability of rolling a sum of 12 is $\frac{1}{12}$. What is the least possible number of faces on the two dice combined?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

Solution

Key Word Logic, Trial and Error

Notice that

$$\begin{aligned} 7 &= 1+6 \\ &\vdots \\ &= 6+1 \end{aligned}$$

are the only cases that could possibly form 7. In another words, regardless of the number of faces for each dice, 6 is the number of cases that could form a rolling sum of 7.

Because the value of the total combination for both are the same, we could infer that the number of cases of having a rolling sum of 10 is $\frac{4}{3} \cdot 6 = 8$. With the number 8, we could also deduce that one dice has 8 sides and the other has at least 9 sides. Thence trial and error could be utilized.

Since 9 is the next smallest number, the case could be tested.

$$\begin{aligned} 12 &= 3+9 \\ &= 4+8 \\ &\vdots \\ &= 8+4 \end{aligned}$$

$\frac{6}{8 \cdot 9} = \frac{1}{12}$ is also true. Therefore, $8 + 9 = \boxed{\textbf{(B)}\ 17}$ $\square$

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