Problem

A, B, C, and D stand in a circle. Simultaneously, each player selects another player at random and points at that person, who must then sit down. What is the probability that A is the only person who remains standing?

Solution

Key Word Counting Strategy

Solution I

$$\begin{aligned}
\text{total cases of only } A \text{ standing} &= \text{total cases of } A \text{ standing} - \text{total cases of } A \text{ and } (B, C, \text{ or } D) \text{ standing} \\
&= \frac{3}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} - \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot {}_3C_2 \\
&= \frac{8}{27} - \frac{4}{27} \\
&= \boxed{\frac{4}{27}}
\end{aligned}$$

Solution II

$$\begin{array}{cccc}
A & B & C & D \\
\hline
B & C & D & B \\
 & & & C \\
 & D & & \\
C & & & \\
D & & &
\end{array}$$

$$\therefore \text{ total cases of only } A \text{ standing} = \frac{3\cdot 2\cdot 1\cdot 2}{3^4} = \boxed{\frac{4}{27}}$$

□

Problem

R is blindfolded and standing 1 step away from an ice cream stand. Every second, he has a $\frac{1}{4}$ probability of walking 1 step towards the ice cream stand, and a $\frac{3}{4}$ probability of walking 1 step away from the ice cream stand. When he is 0 steps away from the ice cream stand, he wins. What is the probability that R eventually wins?

Solution

Key Word Recursion

First and foremost, variables for recursion formula could be set.

$$\begin{gathered}S_1 = \text{Probability to win from 1}\\ S_2 = \text{Probability to win from 2}\\ \vdots\end{gathered}$$

Using the defined variables, recursion equations could be set.

$$\begin{gathered}S_1 = \frac{1}{4} + \frac{3}{4}S_2\\ S_2 = \frac{1}{4}S_1 + \frac{3}{4}S_3\\ S_3 = \frac{1}{4}S_2 + \frac{3}{4}S_4\\ \vdots\\ S_n = \frac{1}{4}S_{n-1} + \frac{3}{4}S_{n+1}\end{gathered}$$

Notice that S_1 is the value that need to be computed since R is located at 1.

$$S_1 + S_2 + \cdots + S_n = \frac{1}{4} + \frac{1}{4}(S_1 + S_2 + \cdots + S_{n-1}) + \frac{3}{4}(S_2 + \cdots + S_{n+1})$$

For simplicity, let $A = S_1 + S_2 + \cdots + S_n$.

$$\begin{aligned}A &= \frac{1}{4} + \frac{1}{4}(A - S_n) + \frac{3}{4}(A - S_1 + S_{n+1})\\ 0 &= 1 - S_n - 3S_1 + 3S_{n+1}\end{aligned}$$

However, notice that $\lim_{n\to\infty} S_n = 0$.

$$\begin{aligned}\therefore\ 0 &= 1 - 3S_1\\ S_1 &= \boxed{\frac{1}{3}}\end{aligned}$$

□

Problem

Enoch randomly hops between 5 leaves, on each turn hopping to one of the other 4 leaves with equal probability. After 5 hops, what is the probability that Enoch has returned to the leaf where he started?

Solution

Key Word Recursion

First and foremost, variables must be set in order to use recursion.

$$h_x = \text{Probability to be on the leaf where he started after } x \text{ hops}$$
$$h_{x-1} = \frac{1}{4}(1 - h_{x-1})$$

Using the formula above, substitution may lead to the correct answer.

$$h_5 = \frac{1}{4}(1 - h_4) = \boxed{\frac{51}{256}}$$
$$h_4 = \frac{1}{4}(1 - h_3) = \frac{13}{64}$$
$$h_3 = \frac{1}{4}(1 - h_2) = \frac{3}{16}$$
$$h_2 = \frac{1}{4}(1 - h_1) = \frac{1}{4}$$
$$h_1 = 0$$

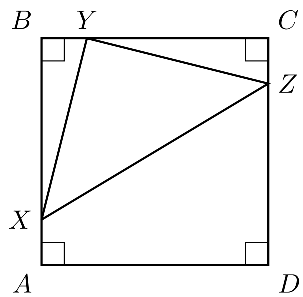
□

Problem

Let $ABCD$ be a unit square and let X, Y, Z be points on sides AB, BC, CD, respectively, such that $AX = BY = CZ$. If the area of triangle XYZ is $\frac{1}{3}$, what is the maximum value of the ratio $\frac{XB}{AX}$?

Solution

Key Word Area Formula



Let $AX = BY = CZ = a$.

$$1 - 2 \cdot \frac{a(1-a)}{2} - \frac{((1-a)+a) \cdot 1}{2} = \frac{1}{3}$$
$$1 - a(1-a) - \frac{1}{2} = \frac{1}{3}$$
$$a = \frac{3 \pm \sqrt{3}}{6}$$

$$\text{The maximum value of } \frac{1}{a} - 1 \text{ must be computed}$$
$$\therefore \max\left(\frac{1}{a} - 1\right) = \boxed{2 + \sqrt{3}}$$

□