# Problem

Four couples are to be seated in a row. If it is required that each woman may only sit next to her husband or another woman, how many different possible seating arrangements are there?

**Solution**

**Key Word** Counting Strategy

**i) One Pair Together**

$w\ w\ w\ w\ m\ m\ m\ m$

$$_4C_1 \cdot 3! \cdot 3! \cdot 2 = \underline{288}$$

**ii) Two Pairs Together**

$w\ w\ w\ m\ m\ m\ m\ w$

$w\ w\ m\ m\ m\ m\ w\ w$

$w\ m\ m\ m\ m\ w\ w\ w$

$$({}_4C_2 \cdot 2 \cdot 2 \cdot 2) \cdot 2 \cdot 3 = \underline{288}$$

**iii) Three Pairs Together**

$w\ w\ m\ m\ w\ w\ m\ m$

$w\ m\ m\ w\ w\ w\ m\ m$

$w\ w\ m\ m\ m\ w\ w\ m$

$w\ m\ m\ m\ w\ w\ w\ m$

$$_4C_3 \cdot 2 \cdot 3! \cdot 4 = \underline{192}$$

**iiii) Four Pairs Together**

$w\ m\ m\ w\ w\ m\ m\ w$

$$4! \cdot 2 = \underline{48}$$

$$288 + 288 + 192 + 48 = \boxed{816}$$

□

# Problem

How many ten-digit positive integers consist of ten different digits and are divisible by 99?

**Solution**

**Key Word** Counting Strategy

Because the sum of all integers from 0 to 9 is 45, only the divisibility rule for 11 must be checked. The pairs of possible sums of alternative digits are $(17, 28)$ and $(6, 39)$. However, because the case $(6, 39)$ is not possible, only the pair $(17, 28)$ must be checked. Because counting for 28 seems easier, let's try counting for 28.

| | | | | |
|---|---|---|---|---|
| 9 | 8 | 6 | 5 | 0 |
| 9 | 8 | 6 | 4 | 1 |
| 9 | 8 | 6 | 3 | 2 |
| 9 | 8 | 5 | 4 | 2 |
| 9 | 7 | 6 | 5 | 1 |
| 9 | 7 | 6 | 4 | 2 |
| 9 | 7 | 5 | 4 | 3 |
| 8 | 7 | 6 | 5 | 2 |
| 8 | 7 | 6 | 4 | 3 |

$$\therefore\ (5! - 4!)5! + 8 \cdot 5!5! + 5!5! + 8(5! - 4!)5! = \boxed{233280}.$$

□

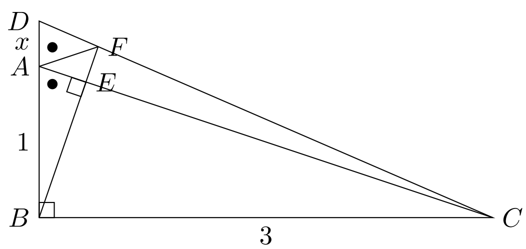
# Problem

$\triangle ABC$ is right-angled at $B$, with $AB = 1$ and $BC = 3$. $E$ is the foot of perpendicular from $B$ to $AC$. $BA$ and $BE$ are produced to $D$ and $F$ respectively such that $D$, $F$, $C$ are collinear and $\angle DAF = \angle BAC$. Find the length of $AD$.

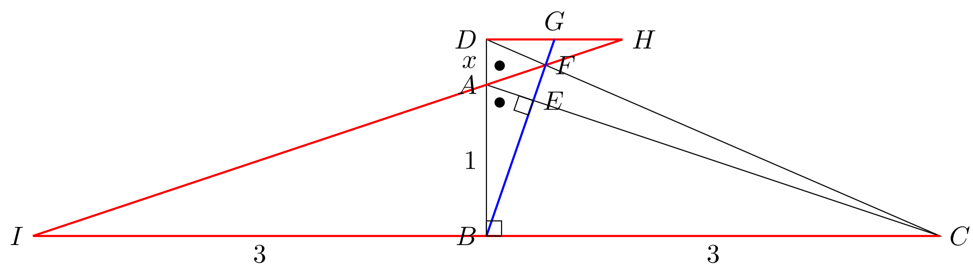
**Solution**

**Key Word** Similar Triangle

First and foremost, let us draw the situation first.




An impulse to extend $BF$ is created. Different extension lines could be drawn to use similar triangles.




Looking carefully at the diagram, it could be inferred that $DG = GH$. The reason is because $\triangle DFS \sim \triangle CFI$ and $IB = BC$. In another words, $DG = \frac{3x}{2}$. Using the similar triangles again, $\frac{3x}{2} : x + 1 = 1 : 3$ is true. Therefore, $x = \boxed{\frac{2}{7}}$. □

# 2021 AMC 12A Problem 12

All the roots of the polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of $B$?

**(A)** $-88$ **(B)** $-80$ **(C)** $-64$ **(D)** $-41$ **(E)** $-40$

**Solution**

**Key Word** Vieta's Formula

Using Vieta's formula, it is true that $r_1 + r_2 + r_3 + r_4 + r_5 + r_6 = 10$ and $r_1r_2r_3r_4r_5r_6 = 16$. Moreover, $r_1r_2r_3 + \cdots + r_4r_5r_6$ is the value that needs to be computed. Remember.. all roots are positive integers!! Thus, WLOG, $r_1 = 1, r_2 = 1, r_3 = 2r_4 = 2, r_5 = 2$ and $r_6 = 2$.

Now, let's be smart. The number of cases of choosing $2, 2, 2$ is ${}_4C_3$. Moreover, the number of cases of selecting $2, 2, 1$ is ${}_4C_2 \cdot {}_2C_1$. Furthermore, the number of cases of choosing $2, 1, 1$ is ${}_4C_1 \cdot {}_2C_2$. Therefore, $8 \cdot {}_4C_3 + 4 \cdot {}_4C_2 \cdot {}_2C_1 + 2 \cdot {}_4C_1 \cdot {}_2C_2 = 32 + 48 + 8 = 88$. We now need to change the sign, since we used Vieta's Formula. Thus, the answer is $\boxed{\textbf{(A)}\ -88}$. □

# 2021 AMC 12A Problem 22

Suppose that the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$, where angles are in radians. What is $abc$?

**(A)** $-\frac{3}{49}$ **(B)** $-\frac{1}{28}$ **(C)** $\frac{\sqrt[3]{7}}{64}$ **(D)** $\frac{1}{32}$ **(E)** $\frac{1}{28}$

**Solution**

**Key Word** Vieta's Formula, The Roots of Unity, Euler's Formula, Trigonometric Identities

First and foremost, using the Vieta's Formula, it is evident that

$$-a = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$$
$$b = \cos\frac{2\pi}{7}\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7}\cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$$
$$-c = \cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$$

**Calculating** $a$
Using the $7^{\text{th}}$ roots of unity, we could know that

$$e^{2\pi\cdot\frac{0}{7}} + e^{2\pi\cdot\frac{1}{7}} + \cdots + e^{2\pi\cdot\frac{5}{7}} + e^{2\pi\cdot\frac{6}{7}}$$

is true. Moreover, using the Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$, it is evident that the following is true.

$$\cos\left(2\pi\cdot\frac{0}{7}\right) + \cdots + \cos\left(2\pi\cdot\frac{6}{7}\right) + i\left(\sin\left(2\pi\cdot\frac{0}{7}\right) + \cdots + \sin\left(2\pi\cdot\frac{6}{7}\right)\right) = 0$$

Using the trigonometric identities, the following equation may be modified.

$$1 + 2\left(\cos\left(2\pi\cdot\frac{1}{7}\right) + \cos\left(2\pi\cdot\frac{2}{7}\right) + \cos\left(2\pi\cdot\frac{3}{7}\right)\right) = 0$$

In another words, $2(-a) = -1$ and $a = \frac{1}{2}$.

**Calculating** $b$
The trigonometric identity $\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$ could be used.

$$\begin{aligned}
b &= \frac{1}{2}\left(\cos\frac{6\pi}{7} + \cos\frac{2\pi}{7}\right) + \frac{1}{2}\left(\cos\frac{8\pi}{7} + \cos\frac{4\pi}{7}\right) + \frac{1}{2}\left(\cos\frac{10\pi}{7} + \cos\frac{2\pi}{7}\right) \\
&= \frac{1}{2}\left(2\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{8\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{10\pi}{7}\right) \\
&= \cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7} \\
&= -a \\
&= -\frac{1}{2}
\end{aligned}$$

**Calculating** $c$

$$
\begin{aligned}
-c = \frac{1}{2}\left(\cos\frac{6\pi}{7}+\cos\frac{2\pi}{7}\right)\left(\cos\frac{6\pi}{7}\right) &= \left(\cos\frac{6\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{6\pi}{7}\right)\\
&= \frac{1}{2}\left(\frac{1}{2}\left(\cos\frac{12\pi}{7}+\cos\frac{0\pi}{7}\right)+\frac{1}{2}\left(\cos\frac{8\pi}{7}+\cos\frac{4\pi}{7}\right)\right)\\
&= \frac{1}{4}\left(1+\cos\frac{12\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)\\
&= \frac{1}{4}(1-a)\\
&= \frac{1}{8}
\end{aligned}
$$

**Final Calculation**

$$
abc = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{1}{8}\right) = \boxed{\textbf{(D)}\ \frac{1}{32}}
$$

□