# 2022 AMC 12B Problem 17

How many $4 \times 4$ arrays whose entries are 0s and 1s are there such that the row sums (the sum of the entries in each row) are $1, 2, 3,$ and $4$, in some order, and the column sums (the sum of the entries in each column) are also $1, 2, 3,$ and $4$, in some order? For example, the array

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

satisfies the condition.

**(A)** 144 **(B)** 240 **(C)** 336 **(D)** 576 **(E)** 624

**Solution**

**Key Word** Counting Strategy

First and Foremost, each column and row could be numbered to find the relationship between each arrangement of columns and rows and the number of possible outcomes.

$$\begin{array}{cccc|c} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ \hline ? & ? & ? & ? & \end{array}$$

From the arrangement above, it is evident that when the 4 and 1 is chosen for some column and row, the possible cases for most 1 and 0 are chosen. The preceding diagram is the example of a case.

$$\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 1 \\ 0 & ? & 1 & ? & ? \\ 0 & ? & 1 & ? & ? \\ 1 & 1 & 1 & 1 & 4 \\ \hline 1 & ? & 4 & ? & \end{array}$$

WLOG, only one possible case of arrangement exists for each arrangement of column and row.

Therefore, the number of possible arrangements is $4! \cdot 4! = \boxed{\textbf{D) } 576}$   □

# 2022 AMC 12A Problem 22

Let $c$ be a real number, and let $z_1$ and $z_2$ be the two complex numbers satisfying the equation $z^2 - cz + 10 = 0$. Points $z_1$, $z_2$, $\frac{1}{z_1}$, and $\frac{1}{z_2}$ are the vertices of (convex) quadrilateral $\mathcal{Q}$ in the complex plane. When the area of $\mathcal{Q}$ obtains its maximum possible value, $c$ is closest to which of the following?

**(A)** $4.5$    **(B)** $5$    **(C)** $5.5$    **(D)** $6$    **(E)** $6.5$

**Solution**

**Key Word** AM-GM Inequality, A Trick on Finding the Maximum Value

First and foremost, because $z_n$ is a complex number, and the value of $z$ could be found in terms of $c$, the given quadratic equation could be solved. WLOG, $z_1 = \frac{c+\sqrt{c^2-40}}{2}$ and $z_1 = \frac{c-\sqrt{c^2-40}}{2}$. If $c^2 - 40 \geq 0$, no quadrilateral will form. Thereby, it could be inferred that $c^2 - 40 < 0$.

Each complex number could be represented as a point.

$$z_1 \left( \frac{c}{2}, \frac{\sqrt{40-c^2}}{2} \right)$$

$$z_2 \left( \frac{c}{2}, \frac{-\sqrt{40-c^2}}{2} \right)$$

$$\frac{1}{z_1} \left( \frac{c}{20}, \frac{c - \sqrt{40-c^2}}{20} \right)$$

$$\frac{1}{z_2} \left( \frac{c}{20}, \frac{c + \sqrt{40-c^2}}{20} \right)$$

Notice that vertices form a trapezoid. Therefore, the area of the trapezoid $\mathcal{Q}$ could be represented as

$$\frac{\left( \frac{c}{2} - \frac{c}{20} \right) \left\{ \left( \frac{\sqrt{40-c^2}}{2} + \frac{\sqrt{40-c^2}}{2} \right) + \left( \frac{c+\sqrt{40-c^2}}{20} - \frac{c-\sqrt{40-c^2}}{20} \right) \right\}}{2}.$$

A RULE OF THUMB: Since we are finding the maximum value, any desired constant could be multiplied. Multiplying 80 and other constants to the equation above may provide a simplified form.

$$(9c)(20\sqrt{40-c^2} + 2\sqrt{40-c^2})$$
$$\Rightarrow 9c(22\sqrt{40-c^2})$$
$$\Rightarrow c\sqrt{40-c^2}$$

PLEASE DON'T USE OPTIMIZATION HERE!! What can we do? We can use AM-GM Inequality! According to the AM-GM Inequality,

$$40 - c^2 + c^2 \geq 2\sqrt{(40-c^2) \cdot c^2}.$$

$\sqrt{(40-c^2) \cdot c^2}$ reaches maximum when $40 - c^2 = c^2$. In another words, $c = \pm 2\sqrt{5}$. Using 2.2 for the approximation of $\sqrt{5}$, $\max(c) \approx 4.4$. Thus, the closest value in the answer choice is $\boxed{\textbf{(A) } 4.5}$.    $\square$

Uploaded a new solution in AOPS!! Also, user name changed from "thinkingtree" to "MaPhyCom"!