2022 AMC 12B Problem 15

One of the following numbers is not divisible by any prime number less than 10. Which is it?

(A) $2^{606} - 1$ **(B)** $2^{606} + 1$ **(C)** $2^{607} - 1$ **(D)** $2^{607} + 1$ **(E)** $2^{607} + 3^{607}$

Solution

Key Word Property of Modular Arithmetic, Euler's Theorem

Choice (A):

$$
\begin{aligned}
2^{606} - 1 &\equiv (-1)^{606} - 1 \pmod 3 \\
&\equiv 0 \pmod 3
\end{aligned}
$$

Choice (B):

$$
\begin{aligned}
2^{\varphi(5)} &\equiv 1 \pmod 5 \\
2^4 &\equiv 1 \pmod 5 \\
2^{604} &\equiv 1 \pmod 5 \\
2^{606} &\equiv 4 \pmod 5 \\
2^{606} + 1 &\equiv 0 \pmod 5
\end{aligned}
$$

Choice (C):

$$
\begin{aligned}
2^{\varphi(3)} &\equiv 1 \pmod 3 \\
2^2 &\equiv 1 \pmod 3 \\
2^{607} &\equiv 2 \pmod 3 \\
2^{607} - 1 &\equiv 1 \pmod 3
\end{aligned}
\qquad
\begin{aligned}
2^{\varphi(5)} &\equiv 1 \pmod 5 \\
2^4 &\equiv 1 \pmod 5 \\
2^{604} &\equiv 1 \pmod 5 \\
2^{607} - 1 &\equiv 7 \pmod 5
\end{aligned}
\qquad
\begin{aligned}
2^{\varphi(7)} &\equiv 1 \pmod 7 \\
2^6 &\equiv 1 \pmod 7 \\
2^{606} &\equiv 1 \pmod 7 \\
2^{607} - 1 &\equiv 1 \pmod 7
\end{aligned}
$$

Therefore, $\boxed{\textbf{(C) } 2^{607} - 1}$ is not divisible by any prime number less than 10. $\qquad\square$

2022 AMC 12B Problem 20

Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial $x^2 + x + 1$, the remainder is $x + 2$, and when $P(x)$ is divided by the polynomial $x^2 + 1$, the remainder is $2x + 1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

(A) 10 **(B)** 13 **(C)** 19 **(D)** 20 **(E)** 23

Solution

Key Word Property of Modular Arithmetic

First and foremost, the provided condition may be written.

$$P(x) \equiv x + 2 \pmod{x^2 + x + 1}$$
$$P(x) \equiv 2x + 1 \pmod{x^2 + 1}$$

The relationship between the first and the second condition could be found.

$$(x^2 + x + 1)k + x + 2 \equiv 2x + 1 \pmod{x^2 + 1}$$
$$(x^2 + x + 1)k \equiv x - 1 \pmod{x^2 + 1}$$
$$(x^2 + x + 1)k \equiv x^2 + x \pmod{x^2 + 1}$$
$$(x^2 + 1)k + xk \equiv x^2 + x \pmod{x^2 + 1}$$
$$xk \equiv x^2 + x \pmod{x^2 + 1}$$
$$k \equiv x + 1 \pmod{x^2 + 1}$$

k could be rewritten as $k = (x^2+1)k'+x+1$. Therefore, $P(x) = (x^2+x+1)(x^2+1)k'+(x^2+x+1)(x+1)+x+2$. The degree of $P(x)$ is minimum when $k' = 0$. Thereby, $P(x) = (x^2+x+1)(x+1)+x+2 = x^3+2x^2+3x+3$. Thus, $1^1 + 2^2 + 3^2 + 3^2 = \boxed{\textbf{E) } 23}$. $\qquad \square$

2022 AMC 12B Problem 23

Let $x_0, x_1, x_2, \ldots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n, define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k$$

Suppose $7S_n \equiv 1 \pmod{2^n}$ for all $n \geq 1$. What is the value of the sum

$$x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022}?$$

(A) 6　　**(B)** 7　　**(C)** 12　　**(D)** 14　　**(E)** 15

Solution

Key Word Trial and Error, Euler's Theorem

Using the given conditions, it is evident that $x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022} = \frac{S_{2023} - S_{2019}}{2^{2019}} = a$ (for simplicity). Moreover, because $7S_n \equiv 1 \pmod{2^n}$ is true, an impulse to multiply the numerator and denominator by 7 is created. Let $7S_{2023} = 2^{2023}k + 1$ and $7S_{2019} = 2^{2019}k' + 1$.

$$\begin{aligned}
a &= \frac{7S_{2023} - 7S_{2019}}{7 \cdot 2^{2019}} = \frac{2^{2023}k + 1 - 2^{2019}k' - 1}{7 \cdot 2^{2019}} \\
&= \frac{2^4 k - k'}{7} \\
&= \frac{16k - k'}{7}
\end{aligned}$$

Furthermore, because $0 \leq x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022} \leq 15$, $0 \leq \frac{16k - k'}{7} \leq 15$. The order pairs (k, k') could be found through trial and error that satisfies conditions $7S_{2023} = 2^{2023}k + 1$ and $7S_{2019} = 2^{2019}k' + 1$. It is evident that $k \neq 0, 1, 2$ using Euler's Theorem.

k	k'	a	Validity
3	6	6	Yes $(\because 7 \mid 2^{2023} \cdot 3 + 1$ and $7 \mid 2^{2019} \cdot 6)$

Thereby, $x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022} = \boxed{\textbf{(A) } 6}$. □

Uploaded a new solution in AOPS!!