2025 Purple Comet! Math Meet Problem 11

Positive integers m, n, and p satisfy

$$m + n + p = 104 \quad \text{and}$$
$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = \frac{1}{4}.$$

Find the greatest possible value of $\max(m, n, p)$.

Solution

Key Word Trial and Error

Multiplying $4mnp$ on both sides for second equation may lead to a similar result to $(m-4)(n-4)(p-4) = mnp - 4(mn + np + pm) + 16(m + n + p) - 64$. The fact that $4(mn + np + pm) = mnp$ could be driven. Therefore, the following equation is true.

$$(m-4)(n-4)(p-4) = 16 \cdot 105 - 64 = 1600$$

WLOG, let $p \geq n \geq m$.
There is a reasonable doubt that $p = 84$. If p is 84, $m + n = 20$ and $\frac{1}{m} + \frac{1}{n} = \frac{20}{84}$. In another words, $m + n = 20$ and $mn = 84$ with $m = 6$ and $n = 14$. Moreover, there seems to be no other possible cases that generates a bigger p value than 84. Therefore, the greatest possible value of $\max(m, n, p) = \boxed{84}$. □

2025 Purple Comet! Math Meet Problem 24

Three distinct real numbers x_1, x_2, and x_3 in the interval $[0, \pi]$ satisfy the equation $\sec(2x) - \sec x = 2$. There are relatively prime positive integers m and n such that

$$\frac{\pi}{x_1 + x_2 + x_3} = \frac{m}{n}.$$

Find $10m + n$.

Solution

Key Word Trigonometric Identities

$\sec 2x - \sec x = 2$ could be converted to $\frac{1}{\cos 2x} - \frac{1}{\cos x} = 2$. $2\cos^2 x - 1$ could be substituted for $\cos 2x$. Because a cubic equation could be written, a general rule of thumb of SUBSTITUTION could be utilized. Therefore, let $a = \cos x$.

$$\begin{aligned}
\frac{1}{2\cos^2 x - 1} - \frac{1}{\cos x} &= 2 \\
\frac{1}{2a^2 - 1} - \frac{1}{a} &= 2 \\
a - (2a^2 - 1) &= 2a(2a^2 - 1) \\
-2a^2 + a + 1 &= 4a^3 - 2a \\
4a^3 + 2a^2 - 3a - 1 &= 0 \\
(a + 1)(4a^2 - 2a - 1) &= 0
\end{aligned}$$

Because $\cos x = -1, \frac{1 \pm \sqrt{5}}{4}$, $x_1 + x_2 + x_3 = \pi + \frac{\pi}{5} + \frac{3\pi}{5} = \frac{9\pi}{5}$ Therefore, $10m + n = \boxed{59}$. □

2025 Purple Comet! Math Meet Problem 19

The equation

$$(3x+1)(4x+1)(6x+1)(12x+1) = 5$$

has a solution of the form $\frac{-p+i\sqrt{q}}{r}$, where p is a prime number, 1 and r are positive integers, and $i = \sqrt{-1}$. Find $p+q+r$.

Solution

Key Word Substitution

$$(3x+1)(4x+1)(6x+1)(12x+1) = (24x^2+10x+1)(36x^2+15x+1) = 5$$

SUBSTITUTION HERE IS AN UNWRITTEN RULE

Let $a = 24x^2+10x+1$ and $36x^2+15x+1 = \frac{3}{2}a - \frac{1}{2}$. Therefore, $a \cdot \left(\frac{3}{2}a - \frac{1}{2}\right) = 5$. In another words, $a = -\frac{5}{3}, 2$.

The first case to check is when $a = -\frac{5}{3}$

$$\begin{aligned}
24x^2+10x+1 &= -\frac{5}{3} \\
72x^2+30x+8 &= 0 \\
x &= \frac{-15 \pm \sqrt{225-72\cdot 8}}{72} \\
&= \frac{-15 \pm \sqrt{225-576}}{72} \\
&= \frac{-15 \pm \sqrt{-351}}{72} \\
&= \frac{-15 \pm 3\sqrt{-39}}{72} \\
&= \frac{-5 \pm \sqrt{39}i}{24}
\end{aligned}$$

The second case to check is when $a = 2$.

$$\begin{aligned}
24x^2+10x+1 &= 2 \\
24x^2+10x-1 &= 0
\end{aligned}$$

No further calculation is necessary because the discriminant is greater than zero. Therefore, $p+q+r = 5+39+24 = \boxed{68}$. □

2023 AMC 12A Problem 25

There is a unique sequence of integers $a_1, a_2, \cdots, a_{2023}$ such that

$$\tan 2023x = \frac{a_1 \tan x + a_3 \tan^3 x + a_5 \tan^5 x + \cdots + a_{2023} \tan^{2023} x}{1 + a_2 \tan^2 x + a_4 \tan^4 x \cdots + a_{2022} \tan^{2022} x}$$

whenever $\tan 2023x$ is defined. What is a_{2023}?

(A) -2023 **(B)** -2022 **(C)** -1 **(D)** 1 **(E)** 2023

Solution

Key Word Trigonometric Identities

It is known that $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$. Because not much evident action could be taken from the problem, manipulation of known knowledge might be beneficial. Arbitrarily substituting x values for α and β may provide a pattern.

$$\begin{aligned} \tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} &= \frac{\frac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \frac{2\tan x}{1-\tan^2 x} \cdot \tan x} \\ &= \frac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x - 2\tan^2 x} \\ &= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \end{aligned}$$

In similar fashion, $\tan 4x$ could be recalculated.

$$\begin{aligned} \tan 4x = \frac{\tan 3x + \tan x}{1 - \tan 3x \tan x} &= \frac{\frac{3\tan x - \tan^3 x}{1-3\tan^2 x} + \tan x}{1 - \frac{3\tan x - \tan^3 x}{1-3\tan^2 x} \cdot \tan x} \\ &= \frac{3\tan x - \tan^3 x + \tan x - 3\tan^3 x}{1 - 3\tan^2 x - 3\tan^2 x + \tan^4 x} \\ &= \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x} \end{aligned}$$

□

Looking at the pattern, it is evident that $a_{2023} = \boxed{-1}$