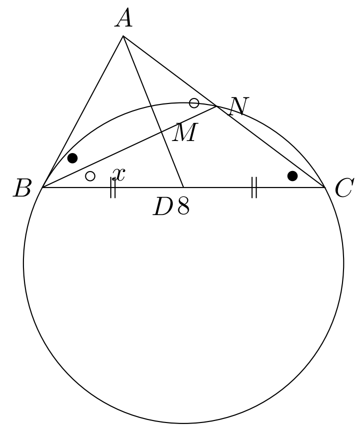
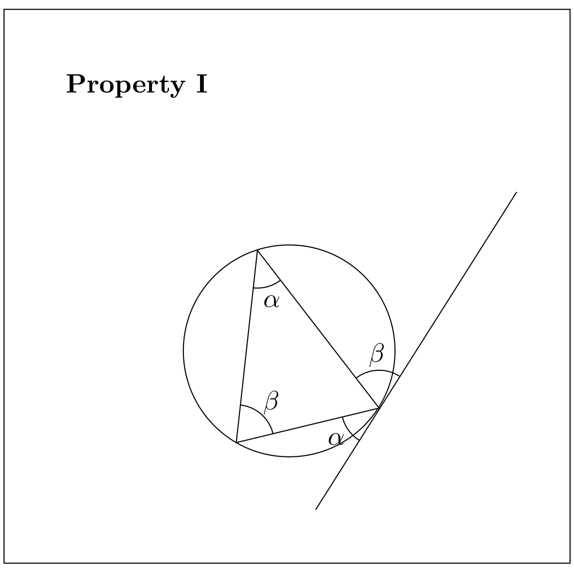
Problem

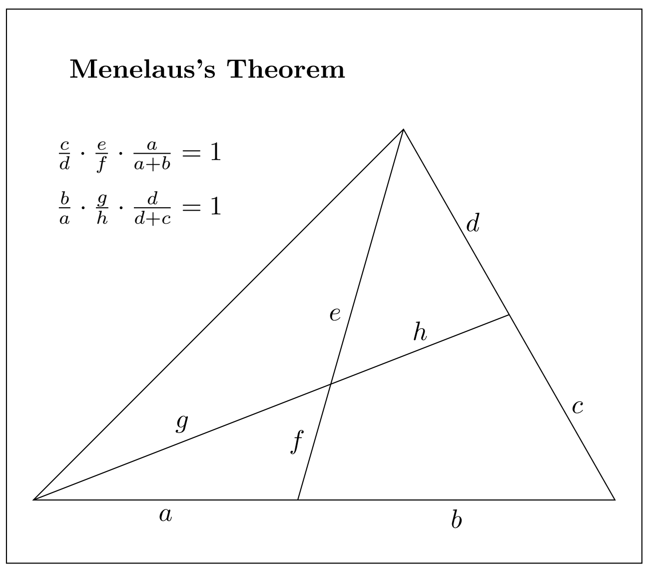
In $\triangle ABC$, D is the mid-point of BC and m is a point of AD. The extension of BM meets AC at N, and AB is tangent to the circumcircle of $\triangle BCN$. If $BC = 8$ and $BN = 6$, find the length of BM.

Solution



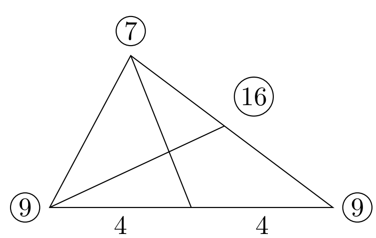
Key Word Mass Point, Menelaus's Theorem, Similar Triangle





Solution I: Mass Point

Using Property I or Power of a Point Theorem, it is evident that $AN : NC : AB = 3 : \frac{7}{3} : 4$. Therefore, masses for each points could be assigned.



$$6 \cdot \frac{16}{16+9} = \boxed{\frac{96}{25}}$$

Solution II: Menelaus's Theorem

Menelaus's Theorem may be utilized to solve the problem.

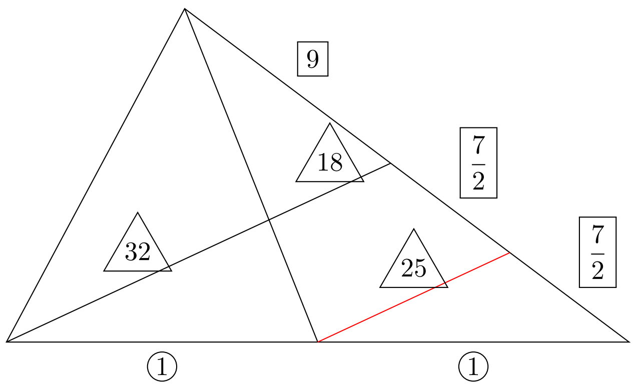
$$\frac{4}{4} \cdot \frac{x}{6-x} \cdot \frac{9}{16} = 1$$

Therefore, x could be calculated.

$$\begin{aligned} \frac{4}{4} \cdot \frac{x}{6-x} \cdot \frac{9}{16} &= 1 \\ 9x &= 16(6-x) \\ x &= \boxed{\frac{96}{25}} \end{aligned}$$

Solution III: Similar Triangle

A parallel line could be drawn to AC from D.



$$BM = 6 \cdot \frac{32}{32+18} = \boxed{\frac{96}{25}}$$

□

Problem

If $a^3 - 3ab^2 = 11$ and $b^3 - 3a^2b = 13$, find the value of $a^2 + b^2$.

Solution

Key Word $\boxed{\text{DO NOT BE AFRAID TO INCREASE DEGREE}}$

I am looking for $a^2 + b^2$. However, in the given equations $a^3 - 3ab^2 = 11$ and $b^3 - 3a^2b = 13$, There are powers with odd numbers. What should I do? I think I want to square them to make the exponents even.

$$\begin{aligned} (a^3 - 3ab^2)^2 &= a^6 - 6a^4b^2 + 9a^2b^4 = 121 \\ (b^3 - 3a^2b)^2 &= b^6 - 6a^2b^4 + 9a^4b^2 = 169 \end{aligned}$$

An impulse to add the equations are created due to like terms.

$$a^6 - 6a^4b^2 + 9a^2b^4 + b^6 - 6a^2b^4 + 9a^4b^2 = a^6 + 3a^4b^2 + 3a^2b^4 + b^6 = (a^2+b^2)^3 = 290$$

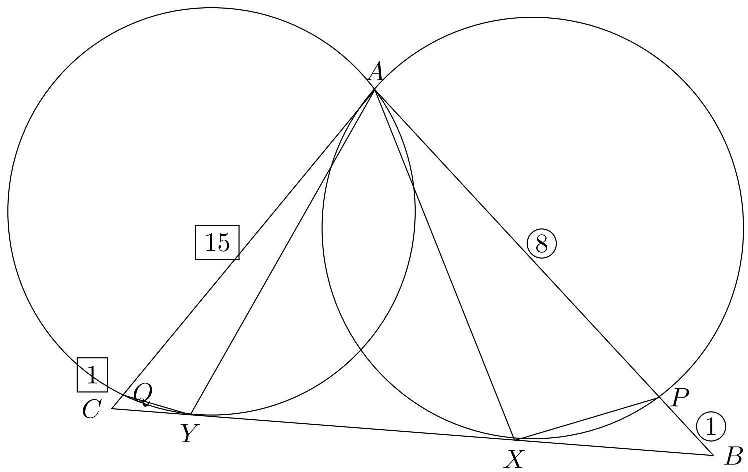
Therefore, $a^2 + b^2 = \boxed{\sqrt[3]{290}}$ □

Problem

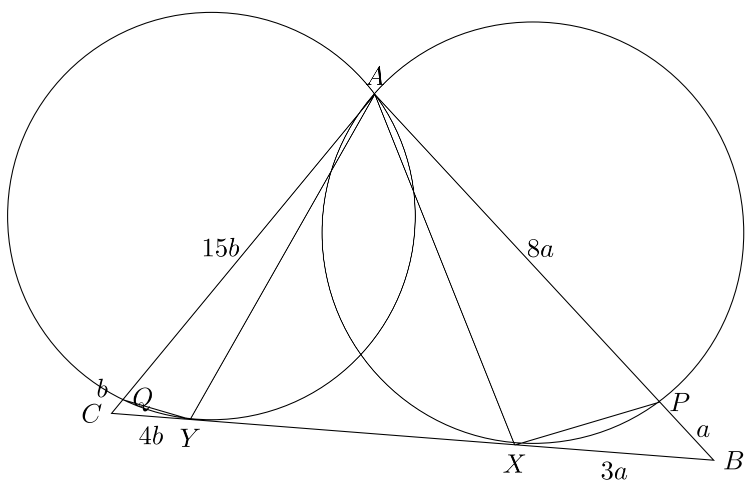
In $\triangle ABC$, P and Q are points of AB and AC respectively such that $AP : PB = 8 : 1$ and $AQ : QC = 15 : 1$. X and Y are points on BC such that the circumcircle of $\triangle APX$ is tangent to both BC and CA, while the circumcircle of $\triangle AQY$ is tangent to both AB and BC. Find $\cos\angle BAC$.

Solution

Key Word Law of Cosines, Power of a Point Theorem



First and foremost, it is evident that $\triangle BPX \sim \triangle BXA$ and $\triangle CQY \sim \triangle CYA$. Thereby, the diagram could utilize different variables.



Because the Law of Cosines utilize ratios, if the ratio between a and b are found, $\cos\angle BAC$ could also be computed.

$AC = CX$ and $AB = BY$ is true. Therefore, $YX = 12b = 6a$.

$$\cos\angle BAC = \frac{8^2 + 9^2 - 11^2}{2 \cdot 8 \cdot 9} = \boxed{\frac{1}{6}}$$

□