

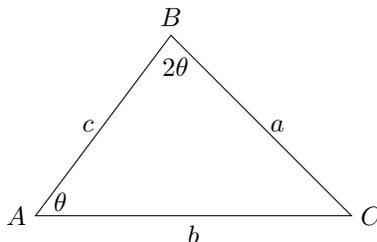
2024 AMC 12B Problem 22

Let $\triangle ABC$ be a triangle with integer side lengths and the property that $\angle B = 2\angle A$. What is the least possible perimeter of such a triangle?

- (A) 13 (B) 14 (C) 15 (D) 16 (E) 17

Solution

Key Word Law of Sines, Law of Cosines, Trigonometric Identities, Property of Triangle



The fact that a, b and c are integers may be utilized.

According to the Law of Sines, the following equation is true.

$$\begin{aligned} \frac{b}{\sin 2\alpha} &= \frac{a}{\sin \alpha} \\ \frac{b}{2 \sin \alpha \cos \alpha} &= \frac{a}{\sin \alpha} \\ \cos \alpha &= \frac{b}{2a} \quad (\because 0^\circ < \alpha < 90^\circ) \end{aligned}$$

Moreover, using the Law of Cosines, an additional equation may be driven.

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

Substitute $\cos \alpha$ to $\cos \alpha = \frac{b}{2a}$.

$$\begin{aligned} \frac{b^2 + c^2 - a^2}{2bc} &= \frac{b}{2a} \\ 2a(b^2 + c^2 - a^2) &= 2b^2c \\ 2ab^2 + 2ac^2 - 2a^3 &= 2b^2c \\ 2b^2(a - c) + 2a(c^2 - a^2) &= 0 \\ 2b^2(a - c) - 2a(a + c)(a - c) &= 0 \\ (a - c)(2b^2 - 2a(a + c)) &= 0 \\ (a - c)(2b^2 - 2a^2 - 2ac) &= 0 \end{aligned}$$

Thus, $a - c = 0$ or $2b^2 - 2a^2 - 2ac = 0$ or both is true. However, when $a = c$, $\alpha = 45^\circ$, which leads to non-integer length for at least one of the sides.

$$\begin{aligned} \therefore 2b^2 - 2a^2 - 2ac &= 0 \\ b^2 &= a^2 + ac \\ b^2 &= a(a + c) \end{aligned}$$

Because the least possible perimeter of such a triangle must be found, the values for a, b and c could be substituted.

b	a	c	Validity
1	1	0	No
2	1	3	No ($\because 3 = 2 + 1$)
2	2	0	No
3	1	8	No ($\because 8 > 3 + 1$)
3	3	0	No
4	1	15	No ($\because 15 > 4 + 1$)
4	2	6	No ($\because 6 = 4 + 2$)
4	4	0	No
5	1	24	No ($\because 24 > 5 + 1$)
6	1	35	No ($\because 35 > 6 + 1$)
6	2	16	No ($\because 16 > 6 + 2$)
6	3	9	No ($\because 9 = 6 + 3$)
6	4	5	Yes

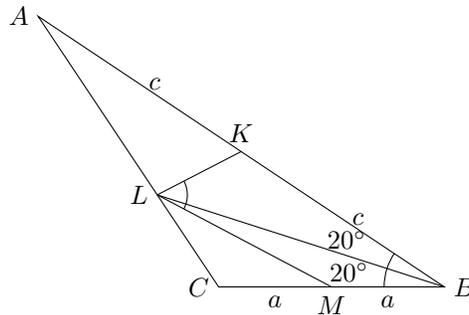
Therefore, the least perimeter is $6 + 4 + 5$, which is (C) 15. □

Problem

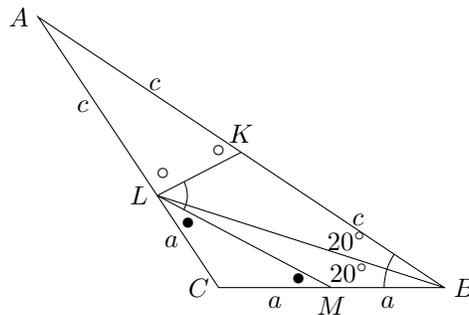
In $\triangle ABC$, $\angle B = 40^\circ$ and $AB + BC = 2AC$. K and M are the mid-points of AB and BC respectively, while L is a point on AC such that BL bisects $\angle ABC$. Find $\angle KLM$.

Solution

Key Word Angle Bisector Theorem



According to the problem, $AC = a + c$. Moreover, using Angle Bisector Theorem, it is evident that $AL = (a + c) \cdot \frac{c}{a+c}$ and $LC = (a + c) \cdot \frac{a}{a+c}$.



Using the property of triangle, $(180 - 2\circ) + (180 - 2\bullet) = 180 - 40$ is true. Thereby, $\circ + \bullet = 110$. In another words, $\angle KLM = 70^\circ$. □

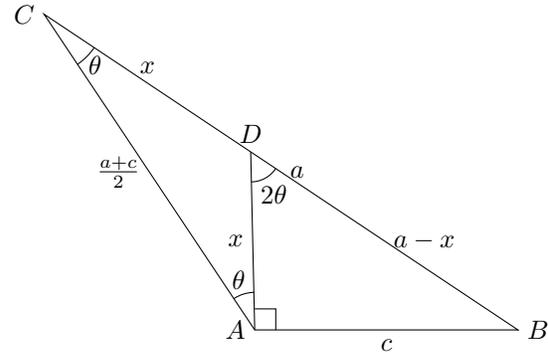
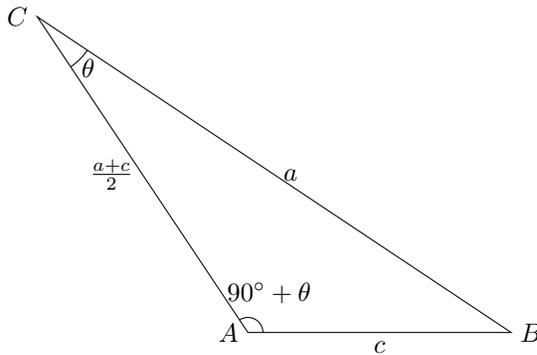
Problem

In $\triangle ABC$, $AB + BC = 2AC$ and $\angle A = \angle C + 90^\circ$. Find $\cos B$.

Solution I

Key Word Law of Sines, Law of Cosines

A vertical line from A could be drawn.



$\cos(90^\circ - 2\theta)$, or $\sin 2\theta$, is the value that is required to be computed. According to the diagram, it is evident that $\sin 2\theta = \frac{c}{a-x}$.

To compute the value of x , trigonometric identities may be utilized. $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ is true.

The Law of Sines may provide additional information.

$$\frac{\frac{a+c}{2}}{\sin(90 - 2\theta)} = \frac{c}{\sin \theta} = \frac{a}{\sin(90 + \theta)}$$

In another words,

$$\begin{aligned} \frac{c}{\sin \theta} &= \frac{a}{\cos \theta} \\ \frac{c}{a} &= \frac{\sin \theta}{\cos \theta} \\ \tan \theta &= \frac{c}{a} \end{aligned}$$

Because $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$, $\frac{c}{x} = \frac{2 \cdot \frac{c}{a}}{1 - (\frac{c}{a})^2}$.

$$\begin{aligned} a - x &= a - c \cdot \frac{1 - (\frac{c}{a})^2}{2 \cdot \frac{c}{a}} \\ &= a - (1 - \frac{c^2}{a^2}) \cdot \frac{c}{2 \cdot \frac{c}{a}} \\ &= a - (1 - \frac{c^2}{a^2}) \cdot \frac{a}{2} \\ &= \frac{a}{2} \left(2 - \left(1 - \frac{c^2}{a^2} \right) \right) \\ &= \frac{a}{2} \left(1 + \frac{c^2}{a^2} \right) \\ &= \frac{a}{2} \left(\frac{a^2 + c^2}{a^2} \right) \\ &= \frac{a^2 + c^2}{2a} \end{aligned}$$

Therefore, the value of $\sin 2\theta = \frac{c}{\frac{a^2+c^2}{2a}} = \frac{2ac}{a^2+c^2}$.

The Law of Cosines may provide supplementary information.

$$\begin{aligned}\cos \theta &= \frac{a^2 + \left(\frac{a+c}{2}\right)^2 - c^2}{2 \cdot a \cdot \frac{a+c}{2}} \\ \cos(90 + \theta) &= \frac{c^2 + \left(\frac{a+c}{2}\right)^2 - a^2}{2 \cdot c \cdot \frac{a+c}{2}} = -\sin \theta\end{aligned}$$

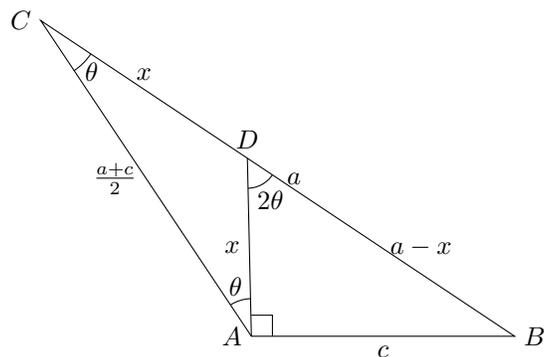
Using the fact $\tan \theta = \frac{c}{a}$ again, the following equations may be written.

$$\begin{aligned}\frac{c}{a} &= \frac{\frac{a^2 - \left(\frac{a+c}{2}\right)^2 - c^2}{2 \cdot c \cdot \frac{a+c}{2}}}{\frac{a^2 + \left(\frac{a+c}{2}\right)^2 - c^2}{2 \cdot a \cdot \frac{a+c}{2}}} \\ &= \frac{\frac{a^2 - \left(\frac{a+c}{2}\right)^2 - c^2}{c}}{\frac{a^2 + \left(\frac{a+c}{2}\right)^2 - c^2}{a}} \\ \frac{c^2}{a^2} &= \frac{a^2 - c^2 - \left(\frac{a+c}{2}\right)^2}{a^2 - c^2 + \left(\frac{a+c}{2}\right)^2} \\ c^2 \left(a^2 - c^2 + \left(\frac{a+c}{2}\right)^2 \right) &= a^2 \left(a^2 - c^2 - \left(\frac{a+c}{2}\right)^2 \right) \\ a^2 c^2 - c^4 + \frac{c^2}{4} (a+c)^2 &= a^4 - a^2 c^2 - \frac{a^2}{4} (a+c)^2 \\ c^2 (a^2 - c^2) + \frac{c^2}{4} (a+c)^2 &= a^2 (a^2 - c^2) - \frac{a^2}{4} (a+c)^2 \\ c^2 (a-c) + \frac{c^2}{4} (a+c) &= a^2 (a-c) - \frac{a^2}{4} (a+c) \\ \frac{c^2}{4} (a+c) + \frac{a^2}{4} (a+c) &= a^2 (a-c) - c^2 (a-c) \\ (a+c) \left(\frac{c^2}{4} + \frac{a^2}{4} \right) &= (a-c) (a^2 - c^2) \\ \left(\frac{c^2}{4} + \frac{a^2}{4} \right) &= (a-c) (a-c) \\ c^2 + a^2 &= 4a^2 - 8ac + 4c^2 \\ 3a^2 + 3c^2 &= 8ac \\ \frac{2ac}{a^2 + c^2} &= \frac{3}{4}\end{aligned}$$

$$\therefore \sin 2\theta = \boxed{\frac{3}{4}}$$

Solution II

Key Word Trigonometric Identities, Law of Sines



From the diagram above, the Law of Sines could be used.

$$\frac{c}{\sin \theta} = \frac{a}{\sin(90 + \theta)} = \frac{\frac{a+c}{2}}{\sin B} = \frac{a+c}{\sin \theta + \sin(90 + \theta)}$$

In another words,

$$\begin{aligned} 2 \sin B &= \sin \theta + \sin(90 + \theta) = 2 \sin \frac{\theta + (\theta + 90)}{2} \cos \frac{\theta - (\theta + 90)}{2} \\ &= 4 \sin \frac{B}{2} \cos \frac{B}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} 4 \sin \frac{B}{2} \cos \frac{B}{2} &= 2 \sin \frac{\theta + (\theta + 90)}{2} \cos \frac{\theta - (\theta + 90)}{2} \\ &= 2 \sin \frac{2\theta + 90}{2} \cos 45^\circ \\ &= \sqrt{2} \cos \frac{B}{2} \\ 4 \sin \frac{B}{2} &= \sqrt{2} \\ \sin \frac{B}{2} &= \frac{\sqrt{2}}{4} \\ \sqrt{\frac{1 - \cos B}{2}} &= \frac{\sqrt{2}}{4} \\ \frac{1 - \cos B}{2} &= \frac{1}{8} \\ \therefore \cos B &= \frac{3}{4} \end{aligned}$$