

Trigonometric Identities

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1 List of Relationships and Formulas

1. $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
2. $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
3. $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$
4. $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$
5. $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$
6. $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$
7. $\sin 2\alpha = 2 \sin \alpha \cos \alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$
8. $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}$
9. $\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$
10. $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$
11. $\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$
12. $\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$
13. $\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$
14. $\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}$
15. $\sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$
16. $\cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$
17. $\tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$
18. $\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$
19. $\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$
20. $\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$
21. $\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$
22. $\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$
23. $\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$
24. $\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$
25. $\sin \alpha \sin \beta = -\frac{1}{2} [\cos(\alpha + \beta) - \cos(\alpha - \beta)]$
26. $\sin^2 \alpha = \frac{1}{2} (1 - \cos 2\alpha)$
27. $\cos^2 \alpha = \frac{1}{2} (1 + \cos 2\alpha)$

2 Angle Sum Relationships

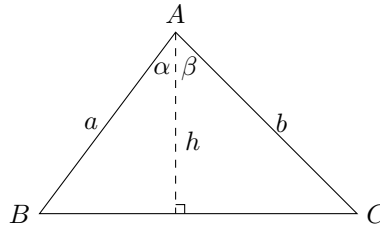
2.1 List of Relationships

$$\begin{aligned}\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}\end{aligned}$$

2.2 Relationship for $\sin(\alpha + \beta)$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

Proof. Consider the diagram below.



It is evident that $h = a \cos \alpha = b \cos \beta$ because $\cos \alpha = \frac{a}{y}$ and $\cos \beta = \frac{b}{y}$. Using the area formula of triangle, the following equation is true.

$$\frac{1}{2}ab \sin(\alpha + \beta) = \frac{1}{2}ah \sin \alpha + \frac{1}{2}bh \sin \beta$$

h could be replaced with $b \cos \beta$ and $a \cos \alpha$ respectively.

$$\frac{1}{2}ab \sin(\alpha + \beta) = \frac{1}{2}ab \cos \beta \sin \alpha + \frac{1}{2}ba \cos \alpha \sin \beta$$

By multiplying both sides by two and dividing by ab (because $ab \neq 0$), the following relationship may be obtained. Moreover, despite the fact that the current proof is not feasible for the case where $ab = 0$, the relationship appears to be true for $ab = 0$ also.

$$\therefore \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

□

2.3 Relationship for $\cos(\alpha + \beta)$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

Proof. Let $\alpha = \frac{\pi}{2} + \gamma$. α could be substituted for $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$. The following equation is obtained.

$$\sin\left(\frac{\pi}{2} + \gamma + \beta\right) = \sin\left(\frac{\pi}{2} + \gamma\right) \cos \beta + \cos\left(\frac{\pi}{2} + \gamma\right) \sin \beta$$

Using the properties of trigonometric functions, $\sin(\frac{\pi}{2} + \gamma) = \cos \gamma$ and $\cos(\frac{\pi}{2} + \gamma) = -\sin \gamma$ are true.

$$\therefore \sin\left(\frac{\pi}{2} + \gamma\right) \cos \beta + \cos\left(\frac{\pi}{2} + \gamma\right) \sin \beta = \cos \gamma \cos \beta - \sin \gamma \sin \beta$$

In another words, $\sin(\frac{\pi}{2} + \gamma + \beta) = \cos(-\gamma - \beta) = \cos(\gamma + \beta) = \cos \gamma \cos \beta - \sin \gamma \sin \beta$ is true. The equation could be rewritten by changing the variables to adhere to convention in using variables:

$$\therefore \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$

□

2.4 Relationship for $\tan(\alpha + \beta)$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

Proof. $\tan(\alpha + \beta)$ could be rewritten as $\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$ by definition. Moreover, because $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ and $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$, it is evident that

$$\frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}$$

is true. $\cos \alpha \cos \beta$ could be factored from both the numerator and the denominator.

$$\frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} = \frac{(\cos \alpha \cos \beta)(\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta})}{(\cos \alpha \cos \beta)(1 - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta})}$$

The terms may be organized with different forms.

$$\frac{(\cos \alpha \cos \beta)(\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta})}{(\cos \alpha \cos \beta)(1 - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta})} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

Therefore, the following expression is true.

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

□

3 Angle Difference Relationships

3.1 List of Relationships

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

3.2 Relationship for $\sin(\alpha - \beta)$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

Proof. The equation $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ is proven to be true.

Substitute $-b$ instead of b to obtain $\sin(\alpha - \beta) = \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$.

$$\therefore \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

□

3.3 Relationship for $\cos(\alpha - \beta)$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Proof. $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ is proven to be true.

Substitute $-b$ instead of b to obtain $\cos(\alpha - \beta) = \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$.

$$\therefore \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

□

3.4 Relationship for $\tan(\alpha - \beta)$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

Proof. $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ is proven to be true.

Substitute $-\beta$ instead of β to obtain $\tan(\alpha - \beta) = \frac{\tan \alpha + \tan(-\beta)}{1 - \tan \alpha \tan(-\beta)} = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$.

$$\therefore \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

□

4 Double Angle Relationships

4.1 List of Relationships

$$\begin{aligned}\sin 2\alpha &= 2 \sin \alpha \cos \alpha \\ &= \frac{2 \tan \alpha}{1 + \tan^2 \alpha}\end{aligned}$$

$$\begin{aligned}\cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2 \cos^2 \alpha - 1 \\ &= 1 - 2 \sin^2 \alpha \\ &= \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}\end{aligned}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

4.2 Relationship for $\sin 2\alpha$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

Proof. Substitute α instead of β in the proven equation: $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$.

$$\sin(\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$$

Therefore, the following equation is true.

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

□

$$\sin 2\alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$$

Proof. $\tan 2\alpha$ could be rewritten to use the formulas that would later be proven.

$$\tan 2\alpha = \frac{\sin(2\alpha)}{\cos(2\alpha)} = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

$\cos 2\alpha$ could be multiplied to both sides.

$$\sin 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \cdot \cos 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \cdot \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}$$

The following equation is resulted.

$$\sin 2\alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$$

□

4.3 Relationship for $\cos 2\alpha$

$$\begin{aligned} \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2 \cos^2 \alpha - 1 \\ &= 1 - 2 \sin^2 \alpha \end{aligned}$$

Proof. Substitute α instead of β in the proven equation: $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$.

$$\cos(\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha$$

Therefore, the following equation is true.

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

Using Pythagorean Trigonometric Identity, the equation could be further modified.

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha$$

□

$$\cos 2\alpha = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}$$

Proof. It is evident that $\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$ is true. The equation could be modified to factor $\cos^2 \alpha$ from both numerator and denominator.

$$\begin{aligned} \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha \\ &= \frac{\cos^2 \alpha - \sin^2 \alpha}{1} \\ &= \frac{\cos^2 \alpha - \sin^2 \alpha}{\cos^2 \alpha + \sin^2 \alpha} \\ &= \frac{(\cos^2 \alpha)(1 - \frac{\sin^2 \alpha}{\cos^2 \alpha})}{(\cos^2 \alpha)(1 + \frac{\sin^2 \alpha}{\cos^2 \alpha})} \\ &= \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} \\ \therefore \cos 2\alpha &= \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} \end{aligned}$$

□

4.4 Relationship for $\tan 2\alpha$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

Proof. Substitute α instead of β in the proven equation: $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$.

$$\tan(\alpha + \alpha) = \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha}$$

Therefore, the following equation is true.

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

□

5 Triple Angle Formulas

5.1 List of Formulas

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$$

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$$

Addition and double angle formulas may be utilized to derive triple angle formulas.

5.2 Formula for $\sin 3\alpha$

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$$

Proof. $\sin 3\alpha$ may be rewritten as $\sin(2\alpha + \alpha)$. $\sin(2\alpha + \alpha) = \sin(2\alpha) \cos \alpha + \cos(2\alpha) \sin \alpha$ could be obtained by using addition relationship. The expression could be further modified through using double angle relationship.

$$\sin(2\alpha) \cos \alpha + \cos(2\alpha) \sin \alpha = 2 \sin \alpha \cos^2 \alpha + (\cos^2 \alpha - \sin^2 \alpha) \sin \alpha$$

$2 \sin \alpha \cos^2 \alpha + (\cos^2 \alpha - \sin^2 \alpha) \sin \alpha = 3 \sin \alpha \cos^2 \alpha - \sin^3 \alpha$ could be obtained through rearrangement of terms. Pythagorean trigonometric identities could be utilized to further simplify the equation by using only $\sin \alpha$.

$$\begin{aligned} 3 \sin \alpha \cos^2 \alpha - \sin^3 \alpha &= 3 \sin \alpha (1 - \sin^2 \alpha) - \sin^3 \alpha \\ &= 3 \sin \alpha - 4 \sin^3 \alpha \end{aligned}$$

Therefore, the following equation is true.

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$$

□

5.3 Formula for $\cos 3\alpha$

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$$

Proof. $\cos \alpha$ may be rewritten as $\cos(2\alpha + \alpha)$. $\cos(2\alpha + \alpha) = \cos(2\alpha) \cos \alpha - \sin(2\alpha) \sin \alpha$ could be obtained by using addition relationship. The expression could be further modified through using double angle relationship.

$$\cos(2\alpha) \cos \alpha - \sin(2\alpha) \sin \alpha = (\cos^2 \alpha - \sin^2 \alpha) \cos \alpha - 2 \sin^2 \alpha \cos \alpha$$

Pythagorean trigonometric identities could be utilized to further simplify the equation by using only $\cos \alpha$.

$$\begin{aligned} (\cos^2 \alpha - \sin^2 \alpha) \cos \alpha - 2 \sin^2 \alpha \cos \alpha &= (2 \cos^2 \alpha - 1) \cos \alpha - 2 \cos \alpha (1 - \cos^2) \\ &= 4 \cos^3 \alpha - 3 \cos \alpha \end{aligned}$$

Therefore, the following equation is true.

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$$

□

6 Half Angle Relationships

6.1 List of Relationships

$$\begin{aligned} \sin \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{2}} \\ \cos \frac{\alpha}{2} &= \pm \sqrt{\frac{1 + \cos \alpha}{2}} \\ \tan \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \\ &= \frac{\sin \alpha}{1 + \cos \alpha} \\ &= \frac{1 - \cos \alpha}{\sin \alpha} \end{aligned}$$

$$\begin{aligned} \sin \theta &= \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \\ \cos \theta &= \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \\ \tan \theta &= \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}} \end{aligned}$$

6.2 Relationship for $\sin \frac{\alpha}{2}$

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

Proof. $\frac{\alpha}{2}$ could be substituted instead of α in the proven equation: $\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)$. $\sin^2 \frac{\alpha}{2} = \frac{1}{2}(1 - \cos \alpha)$ could be driven.

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

□

6.3 Relationship for $\cos \frac{\alpha}{2}$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

Proof. $\frac{\alpha}{2}$ could be substituted instead of α in the proven equation: $\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$. $\cos^2 \frac{\alpha}{2} = \frac{1}{2}(1 + \cos \alpha)$ could be driven.

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

□

6.4 Relationship for $\tan \frac{\alpha}{2}$

$$\begin{aligned} \tan \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \\ &= \frac{\sin \alpha}{1 + \cos \alpha} \\ &= \frac{1 - \cos \alpha}{\sin \alpha} \end{aligned}$$

Proof. $\tan \frac{\alpha}{2}$ could be rewritten as $\frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}}$, which is equal to $\pm \frac{\sqrt{\frac{1 - \cos \alpha}{2}}}{\sqrt{\frac{1 + \cos \alpha}{2}}}$. The equation could be further modified.

$$\begin{aligned} \pm \frac{\sqrt{\frac{1 - \cos \alpha}{2}}}{\sqrt{\frac{1 + \cos \alpha}{2}}} &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \pm \sqrt{\frac{(1 - \cos \alpha)(1 + \cos \alpha)}{(1 + \cos \alpha)(1 + \cos \alpha)}} = \pm \sqrt{\frac{\sin^2 \alpha}{(1 + \cos \alpha)^2}} = \frac{\sin \alpha}{1 + \cos \alpha} \\ &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \pm \sqrt{\frac{(1 - \cos \alpha)(1 - \cos \alpha)}{(1 + \cos \alpha)(1 - \cos \alpha)}} = \pm \sqrt{\frac{(1 - \cos \alpha)^2}{\sin^2 \alpha}} = \frac{1 - \cos \alpha}{\sin \alpha} \end{aligned}$$

Therefore, the following relationships are true.

$$\begin{aligned} \tan \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \\ &= \frac{\sin \alpha}{1 + \cos \alpha} \\ &= \frac{1 - \cos \alpha}{\sin \alpha} \end{aligned}$$

□

6.5 Relationship for $\sin \theta$

$$\sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

Proof. From the double angle relationship $\sin(2\alpha) = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$, substitute α for 2α .

$$\therefore \sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

□

6.6 Relationship for $\cos \theta$

$$\cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

Proof. From the double angle relationship $\cos(2\alpha) = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}$, substitute α for 2α .

$$\therefore \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

□

6.7 Relationship for $\tan \theta$

$$\tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

Proof. From the double angle relationship $\tan(2\alpha) = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$, substitute α for 2α .

$$\therefore \tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

□

7 Function Sum and Difference Relationships

7.1 List of Relationships

$$\begin{aligned}\sin \alpha + \sin \beta &= 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \\ \sin \alpha - \sin \beta &= 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \\ \cos \alpha + \cos \beta &= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \\ \cos \alpha - \cos \beta &= -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}\end{aligned}$$

Let $\alpha = \frac{x+y}{2}$ and $\beta = \frac{x-y}{2}$. $\alpha + \beta = x$ and $\alpha - \beta = y$ could be derived.

7.2 Relationship for $\sin \alpha + \sin \beta$

Proof. From $\sin x + \sin y = \sin(\alpha + \beta) + \sin(\alpha - \beta)$, angle sum and difference relationship could be utilized.

$$\begin{aligned}\sin(\alpha + \beta) + \sin(\alpha - \beta) &= (\sin \alpha \cos \beta + \cos \alpha \sin \beta) + (\sin \alpha \cos \beta - \cos \alpha \sin \beta) \\ &= 2 \sin \alpha \cos \beta \\ &= 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}\end{aligned}$$

Therefore, the following expression is true.

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

□

7.3 Relationship for $\sin \alpha - \sin \beta$

Proof. From $\sin x - \sin y = \sin(\alpha + \beta) - \sin(\alpha - \beta)$, angle sum and difference relationship could be utilized.

$$\begin{aligned}\sin(\alpha + \beta) - \sin(\alpha - \beta) &= (\sin \alpha \cos \beta + \cos \alpha \sin \beta) - (\sin \alpha \cos \beta - \cos \alpha \sin \beta) \\ &= 2 \cos \alpha \sin \beta \\ &= 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}\end{aligned}$$

Therefore, the following expression is true.

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

□

7.4 Relationship for $\cos \alpha + \cos \beta$

Proof. From $\cos x + \cos y = \cos(\alpha + \beta) + \cos(\alpha - \beta)$, angle sum and difference relationship could be utilized.

$$\begin{aligned}\cos(\alpha + \beta) + \cos(\alpha - \beta) &= (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + (\cos \alpha \cos \beta + \sin \alpha \sin \beta) \\ &= 2 \cos \alpha \cos \beta \\ &= 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}\end{aligned}$$

Therefore, the following expression is true.

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

□

7.5 Relationship for $\cos \alpha - \cos \beta$

Proof. From $\cos x - \cos y = \cos(\alpha + \beta) - \cos(\alpha - \beta)$, angle sum and difference relationship could be utilized.

$$\begin{aligned}\cos(\alpha + \beta) - \cos(\alpha - \beta) &= (\cos \alpha \cos \beta - \sin \alpha \sin \beta) - (\cos \alpha \cos \beta + \sin \alpha \sin \beta) \\ &= -2 \sin \alpha \sin \beta \\ &= -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}\end{aligned}$$

Therefore, the following expression is true.

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

□

8 Function Product Relationships

8.1 List of Relationships

$$\begin{aligned}\sin \alpha \cos \beta &= \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] \\ \cos \alpha \sin \beta &= \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)] \\ \cos \alpha \cos \beta &= \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)] \\ \sin \alpha \sin \beta &= -\frac{1}{2} [\cos(\alpha + \beta) - \cos(\alpha - \beta)]\end{aligned}$$

Function sum and difference relationships may be utilized to prove the function product relationships.

Let $\alpha = \frac{x+y}{2}$ and $\beta = \frac{x-y}{2}$. $\alpha + \beta = x$ and $\alpha - \beta = y$ could be derived.

8.2 Relationship for $\sin \alpha \cos \beta$

Proof. It is evident that the following equation is true.

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

The equation may be reorganized in terms of α and β .

$$\begin{aligned} \sin x + \sin y &= 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} \\ \sin(\alpha + \beta) + \sin(\alpha - \beta) &= 2 \sin \alpha \cos \beta \\ \therefore \sin \alpha \cos \beta &= \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] \end{aligned}$$

□

8.3 Relationship for $\cos \alpha \sin \beta$

Proof. It is evident that the following equation is true.

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

The equation may be reorganized in terms of α and β .

$$\begin{aligned} \sin x - \sin y &= 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2} \\ \sin(\alpha + \beta) - \sin(\alpha - \beta) &= 2 \cos \alpha \sin \beta \\ \therefore \cos \alpha \sin \beta &= \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)] \end{aligned}$$

□

8.4 Relationship for $\cos \alpha \cos \beta$

Proof. It is evident that the following equation is true.

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

The equation may be reorganized in terms of α and β .

$$\begin{aligned} \cos x + \cos y &= 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2} \\ \cos(\alpha + \beta) + \cos(\alpha - \beta) &= 2 \cos \alpha \cos \beta \\ \therefore \cos \alpha \cos \beta &= \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)] \end{aligned}$$

□

8.5 Relationship for $\sin \alpha \sin \beta$

Proof. It is evident that the following equation is true.

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

The equation may be reorganized in terms of α and β .

$$\begin{aligned}\cos x - \cos y &= 2 \sin \frac{x+y}{2} \sin \frac{x-y}{2} \\ \cos(\alpha + \beta) - \cos(\alpha - \beta) &= 2 \sin \alpha \sin \beta \\ \therefore \sin \alpha \sin \beta &= \frac{1}{2} [\cos(\alpha + \beta) - \cos(\alpha - \beta)]\end{aligned}$$

□

9 Power Reduction Formulas

9.1 List of Formulas

$$\begin{aligned}\sin^2 \alpha &= \frac{1}{2}(1 - \cos 2\alpha) \\ \cos^2 \alpha &= \frac{1}{2}(1 + \cos 2\alpha)\end{aligned}$$

9.2 Formula for $\sin^2 \alpha$

Proof. Because $\cos(2\alpha) = 1 - 2\sin^2 \alpha$ is proven to be true, the equation could be rearranged to reduce the power for $\sin^2 \alpha$.

$$\therefore \sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)$$

□

9.3 Formula for $\cos^2 \alpha$

Proof. Because $\cos(2\alpha) = 2\cos^2 \alpha - 1$ is proven to be true, the equation could be rearranged to reduce the power for $\cos^2 \alpha$.

$$\therefore \cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$$

□

10 Example

10.1 Example I: Compute $\cos 10^\circ \cdot \cos 30^\circ \cdot \cos 50^\circ \cdot \cos 70^\circ$.

Solution Using function product relationship and known trigonometric identity, the equation could be

rewritten.

$$\begin{aligned}
 \cos 10^\circ \cdot \cos 30^\circ \cdot \cos 50^\circ \cdot \cos 70^\circ &= \cos 10^\circ \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} [\cos 50^\circ \cdot \cos 70^\circ] \\
 &= \cos 10^\circ \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} [\cos(50^\circ + 70^\circ) + \cos(50^\circ - 70^\circ)] \\
 &= \frac{\sqrt{3}}{4} \cos 10^\circ \cdot [\cos 120^\circ + \cos 20^\circ] \\
 &= \frac{\sqrt{3}}{4} \cos 10^\circ \cdot \left[-\frac{1}{2} + \cos 20^\circ\right] \\
 &= -\frac{\sqrt{3}}{8} \cos 10^\circ + \frac{\sqrt{3}}{4} \cos 10^\circ \cos 20^\circ \\
 &= -\frac{\sqrt{3}}{8} \cos 10^\circ + \frac{\sqrt{3}}{4} \cdot \frac{1}{2} [\cos 30^\circ + \cos 10^\circ] \\
 &= -\frac{\sqrt{3}}{8} \cos 10^\circ + \frac{\sqrt{3}}{4} \cdot \frac{1}{2} \left[\frac{\sqrt{3}}{2} + \cos 10^\circ\right] \\
 &= -\frac{\sqrt{3}}{8} \cos 10^\circ + \frac{3}{16} + \frac{\sqrt{3}}{8} \cos 10^\circ \\
 &= \boxed{\frac{3}{16}}
 \end{aligned}$$

□

10.2 Example II: Compute $\sin 18^\circ$.

Solution Let $x = 18^\circ$

$$\begin{aligned}
 5x &= 90^\circ \\
 2x + 3x &= 90^\circ \\
 2x &= 90^\circ - 3x
 \end{aligned}$$

Therefore, $\sin 2x = \sin(90^\circ - 3x) = \cos 3x$. In another terms, the following is true.

$$2 \sin x \cos x = 4 \cos^3 x - 3 \cos x$$

Because $\cos x$ is not zero, the following could be yielded.

$$\begin{aligned}
 2 \sin x &= 4 \cos^2 x - 3 \\
 2 \sin x &= 4(1 - \sin^2 x) - 3 \\
 4 \sin^2 x + 2 \sin x - 1 &= 0
 \end{aligned}$$

Let $y = \sin x$. Therefore, $4y^2 + 2y - 1 = 0$.

$$y = \frac{-1 \pm \sqrt{5}}{4}$$

However, $y = \frac{-1 + \sqrt{5}}{4}$ because $\sin 18^\circ$ is greater than zero.

$$\therefore \sin 18^\circ = \boxed{\frac{-1 + \sqrt{5}}{4}}$$

□