Ancient File 3

Enoch Yu

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Hello! My math group and I made bunch of fun problems in the past, and I wrote solutions for some problems. Apparently, I lost the problem sets and only have the solutions :(Here's what I have!

Problem 1

Key Word: $\frac{1}{n(n+k)} = \frac{1}{k}(\frac{1}{n} - \frac{1}{n+k})$

$$
\begin{aligned}
&\frac{1}{3} - \frac{1}{4} + \frac{1}{15} - \frac{1}{12} + \cdots + \frac{1}{1155} - \frac{1}{612} + \frac{1}{1295} - \frac{1}{684} \\
=&\frac{1}{3} + \frac{1}{15} + \cdots + \frac{1}{1155} + \frac{1}{1295} - \frac{1}{4} - \frac{1}{12} \cdots - \frac{1}{612} - \frac{1}{684} \\
=&\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{33\cdot 35} + \frac{1}{35\cdot 37} - \frac{2}{2\cdot 4} - \frac{2}{4\cdot 6} \cdots - \frac{2}{34\cdot 36} - \frac{2}{36\cdot 38} \\
=&\frac{1}{2}(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{33} - \frac{1}{35} + \frac{1}{35} - \frac{1}{37}) - 2[\frac{1}{2}(\frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \cdots + \frac{1}{34} - \frac{1}{36} + \frac{1}{36} - \frac{1}{38})] \\
=&\frac{1}{2}(\frac{1}{1} - \frac{1}{37}) - 2[\frac{1}{2}(\frac{1}{2} - \frac{1}{38})] \\
=&\frac{1}{2}(\frac{1}{1} - \frac{1}{37}) - (\frac{1}{2} - \frac{1}{38}) \\
=&\frac{1}{2} \cdot \frac{36}{37} - (\frac{1}{2} - \frac{1}{38}) \\
=&\frac{18}{37} - \frac{18}{38} \\
=&\frac{18(38-37)}{37\cdot 38} \\
=&\frac{18}{37\cdot 38} \\
=&\frac{9}{703}
\end{aligned}
$$

Conclusion: **The computed value is $\frac{9}{703}$.**

Problem 5

Key Word: Similar Triangle

First and foremost, the given ratios of diverse length may be altered with least common multiple to avoid fraction.

$LCM(4+3, 3+1, 5+3) = LCM(7, 4, 8) = 56$

Thereby, $\overline{AD} : \overline{DC} = 4 : 3 = 32 : 24, \overline{CF} : \overline{FB} = 3 : 1 = 42 : 14$ and $\overline{BE} : \overline{EA} = 5 : 3 = 35 : 21$.

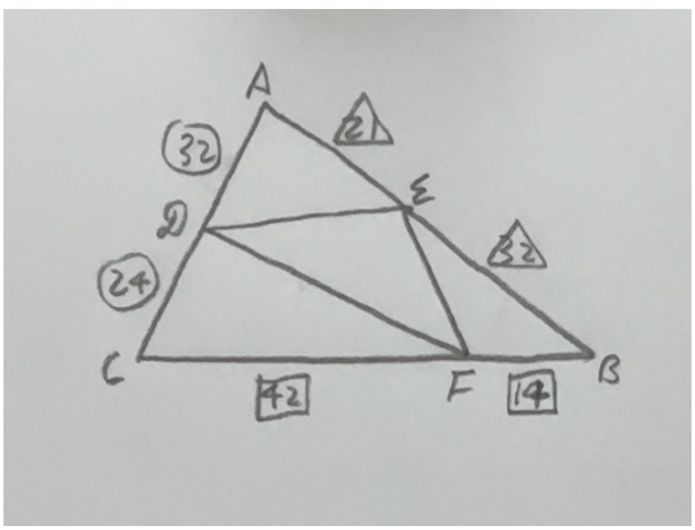
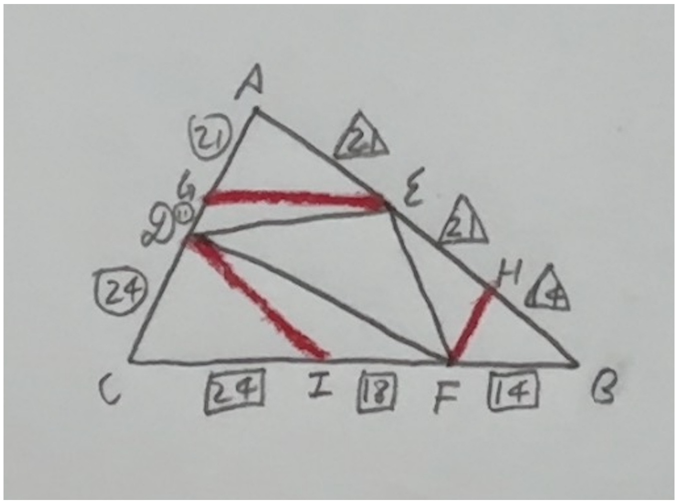


Figure 1: The circles, triangles and rectangles indicate that the numbers enclosed are ratios.

To exploit the property of similar triangles, the three lines parallel to the sides (colored in red) may be drawn.



Using the property of similar triangles, more specified ratios of the lengths are feasibly found.

$$\overline{AE} : \overline{EB} = \overline{AG} : \overline{GC} = 21 : 35 \ (\because \triangle AGE \sim \triangle ACB)$$

$$\overline{AG} : \overline{GD} : \overline{DC} = 21 : 11 : 24$$

$$\overline{CD} : \overline{DA} = \overline{CI} : \overline{IB} = 24 : 35 \ (\because \triangle CDI \sim \triangle CAB)$$

$$\overline{CI} : \overline{IF} : \overline{FB} = 24 : 18 : 14$$

$$\overline{BF} : \overline{FC} = \overline{BH} : \overline{HA} = 14 : 42 \ (\because \triangle BFH \sim \triangle BCA)$$

$$\overline{BH} : \overline{HE} : \overline{EA} = 14 : 21 : 21$$

If x is the area of $\triangle ABC$, the following diagram manifests the specific areas within the triangle.

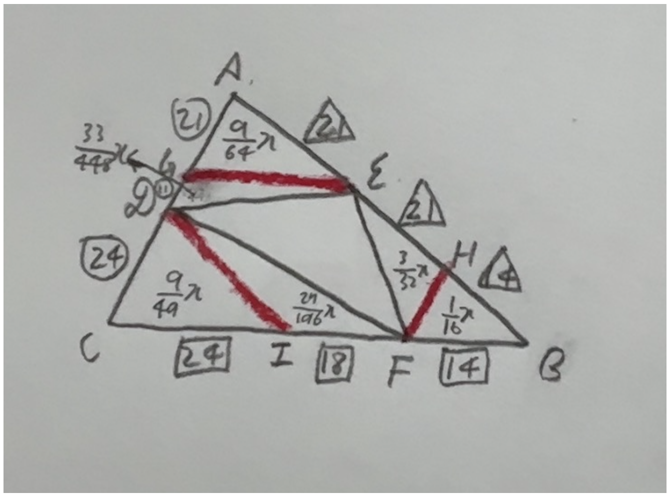


Figure 2: Using the properties of similar triangle, the specific areas within $\triangle ABC$ may be found.

$$
\begin{aligned}
&\frac{9}{64}x + \frac{33}{448}x : \frac{9}{49}x + \frac{27}{196}x : \frac{3}{32}x + \frac{1}{16}x \\
=&\frac{63}{448}x + \frac{33}{448}x : \frac{36}{196}x + \frac{27}{196}x : \frac{3}{32}x + \frac{2}{32}x \\
=&\frac{96}{448} : \frac{63}{196} : \frac{5}{32} \\
=&\frac{3}{14} : \frac{9}{28} : \frac{5}{32} \\
=&\frac{48}{14 \cdot 16} : \frac{72}{28 \cdot 8} : \frac{35}{32 \cdot 7} \\
=&48 : 72 : 35
\end{aligned}
$$

Conclusion: $48 : 72 : 35$ **is the ratio of the areas of** $\triangle ADE, \triangle DCF,$ **and** $\triangle BEF$.

Problem 7

Key Word: $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$

$$
\begin{aligned}
k &= \frac{ab}{2bc-3a+1} = \frac{bc}{2cd-3b-1} = \frac{cd}{2da-3c+1} = \frac{da}{2ab-3d-1} \\
&= \frac{(ab)+(bc)+(cd)+(da)}{(2bc-3a+1)+(2cd-3b-1)+(2da-3c+1)+(2ab-3d-1)} \\
&= \frac{ab+bc+cd+da}{2ab+2bc+2cd+2da-3a-3b-3c-3d+1-1+1-1} \\
&= \frac{ab+bc+cd+da}{2(ab+bc+cd+da)-3(a+b+c+d)} \\
&= \frac{ab+bc+cd+da}{2(ab+bc+cd+da)-3(0)} \\
&= \frac{ab+bc+cd+da}{2(ab+bc+cd+da)} \\
&= \frac{1}{2} \; (\because ab+bc+cd+da \neq 0)
\end{aligned}
$$

Conclusion: **The value of k is $\frac{1}{2}$.**