

Ancient File 2

Enoch Yu

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Hello! My math group and I made bunch of fun problems in the past, and I wrote solutions for some problems. Apparently, I lost the problem sets and only have the solutions :(Here's what I have!

Problem 3

Key Word: Root of Unity, Euler's Formula, Vieta's Formula

Case I: $x^{13} = 1$

According to the Root of Unity, the values of x may be rewritten. Moreover, according to Vieta's Theorem, because the sum of all x is $-\frac{0}{1}$, the sum of all x is 0.

$$\begin{aligned}x_1 &= e^{\frac{0}{13}\pi i} = \cos \frac{0\pi}{13} + i \sin \frac{0\pi}{13} \\+ \\x_2 &= e^{\frac{2}{13}\pi i} = \cos \frac{2\pi}{13} + i \sin \frac{2\pi}{13} \\+ \\x_3 &= e^{\frac{4}{13}\pi i} = \cos \frac{4\pi}{13} + i \sin \frac{4\pi}{13} \\&\vdots \\x_{11} &= e^{\frac{20}{13}\pi i} = \cos \frac{20\pi}{13} + i \sin \frac{20\pi}{13} \\+ \\x_{12} &= e^{\frac{22}{13}\pi i} = \cos \frac{22\pi}{13} + i \sin \frac{22\pi}{13} \\+ \\x_{13} &= e^{\frac{24}{13}\pi i} = \cos \frac{24\pi}{13} + i \sin \frac{24\pi}{13} \\|| \\0\end{aligned}$$

The properties of Sine and Cosine functions states that the following equations satisfy.

$$\begin{array}{llll} \cos \frac{2\pi}{13} & = & \cos \frac{24\pi}{13} & i \sin \frac{2\pi}{13} = -i \sin \frac{24\pi}{13} \\ \cos \frac{4\pi}{13} & = & \cos \frac{22\pi}{13} & i \sin \frac{4\pi}{13} = -i \sin \frac{22\pi}{13} \\ \vdots & = & \vdots & \vdots = \vdots \\ \cos \frac{10\pi}{13} & = & \cos \frac{16\pi}{13} & i \sin \frac{10\pi}{13} = -i \sin \frac{16\pi}{13} \\ \cos \frac{12\pi}{13} & = & \cos \frac{14\pi}{13} & i \sin \frac{12\pi}{13} = -i \sin \frac{14\pi}{13} \end{array}$$

Therefore, the sum of solutions for the equation $x^{13} = 1$ may be rewritten.

$$\begin{aligned}
 & x_1 + x_2 + \cdots + x_{12} + x_{13} \\
 &= \cos \frac{0\pi}{13} + i \sin \frac{0\pi}{13} + \cos \frac{2\pi}{13} + i \sin \frac{2\pi}{13} + \cdots + \cos \frac{22\pi}{13} + i \sin \frac{22\pi}{13} + \cos \frac{24\pi}{13} + i \sin \frac{24\pi}{13} \\
 &= \cos \frac{0\pi}{13} + i \sin \frac{0\pi}{13} + \left(\cos \frac{2\pi}{13} + \cdots + \cos \frac{24\pi}{13} \right) + \left(i \sin \frac{2\pi}{13} + \cdots + i \sin \frac{24\pi}{13} \right) \\
 &= \cos 0 + i \sin 0 + 2 \left(\cos \frac{14\pi}{13} + \cdots + \cos \frac{24\pi}{13} \right) + (0) \\
 &= 1 + 0 + 2 \sum_{k=1}^6 \cos \frac{(2k+12)\pi}{13} + 0 \\
 &= 1 + 2 \sum_{k=1}^6 \cos \frac{(2k+12)\pi}{13}
 \end{aligned}$$

$$\begin{aligned}
 1 + 2 \sum_{k=1}^6 \cos \frac{(2k+12)\pi}{13} &= 0 \\
 \sum_{k=1}^6 \cos \frac{(2k+12)\pi}{13} &= -\frac{1}{2}
 \end{aligned}$$

Therefore, the sum of solutions for the equation $x^{13} = 1$ may be rewritten.

$$\begin{aligned}
 & x_1 + x_2 + \cdots + x_{12} + x_{13} \\
 &= \cos \frac{1\pi}{13} + i \sin \frac{1\pi}{13} + \cos \frac{3\pi}{13} + i \sin \frac{3\pi}{13} + \cdots + \cos \frac{23\pi}{13} + i \sin \frac{23\pi}{13} + \cos \frac{25\pi}{13} + i \sin \frac{25\pi}{13} \\
 &= \cos \frac{13\pi}{13} + i \sin \frac{13\pi}{13} + \left(\cos \frac{1\pi}{13} + \cdots + \cos \frac{11\pi}{13} + \cos \frac{15\pi}{13} + \cdots + \cos \frac{25\pi}{13} \right) + \\
 & \quad \left(i \sin \frac{1\pi}{13} + \cdots + i \sin \frac{11\pi}{13} + i \sin \frac{15\pi}{13} + \cdots + i \sin \frac{25\pi}{13} \right) \\
 &= \cos \pi + i \sin \pi + 2 \left(\cos \frac{1\pi}{13} + \cdots + \cos \frac{11\pi}{13} \right) + (0) \\
 &= -1 + 0 + 2 \sum_{k=1}^6 \cos \frac{(2k-1)\pi}{13} + 0 \\
 &= -1 + 2 \sum_{k=1}^6 \cos \frac{(2k-1)\pi}{13}
 \end{aligned}$$

$$\begin{aligned}
 -1 + 2 \sum_{k=1}^6 \cos \frac{(2k-1)\pi}{13} &= 0 \\
 \sum_{k=1}^6 \cos \frac{(2k-1)\pi}{13} &= \frac{1}{2}
 \end{aligned}$$

Final Calculation

$$\begin{aligned}
 \sum_{k=1}^6 \cos \frac{(2k+12)\pi}{13} &= -\frac{1}{2} \\
 \sum_{k=1}^6 \cos \frac{(2k-1)\pi}{13} &= \frac{1}{2} \\
 \sum_{k=1}^6 \cos \frac{(2k+12)\pi}{13} \cdot \sum_{k=1}^6 \cos \frac{(2k-1)\pi}{13} &= -\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4}
 \end{aligned}$$

Conclusion: $-\frac{1}{4}$ is the calculated value.

Problem 4

Key Word: Pigeon Hole Principle

Let D_n be the accumulated number of completed homework problems on n^{th} day.

Because students must solve at least one problem each day and 100 in total, the following expression is feasible.

$$1 \leq D_1 < D_2 < \dots < D_{64} < D_{65} = 100 \quad (1)$$

A positive integer x can be arbitrarily added in the expression to provide the following:

$$1 + x \leq D_1 + x < D_2 + x < \dots < D_{64} + x < D_{65} + x = 100 + x. \quad (2)$$

The expressions (1) and (2) may be combined.

$$1 \leq D_1, D_2, \dots, D_{64}, D_{65}, D_1 + x, D_2 + x, \dots, D_{64} + x < D_{65} + x = 100 + x \quad (3)$$

According to expression (1), it is evident that $D_1, D_2, \dots, D_{64}, D_{65}$ are all different. Similarly, $D_1 + x, D_2 + x, \dots, D_{64} + x, D_{65} + x$ are all distinct according to expressions (2).

There exists 130 pigeons for expression (3) and retains $100 + x$ pigeon holes. According to the Pigeon Hole Principle, if the number of pigeons exceeds the number of pigeon holes, at least two pigeons must share a hole. In another term, if $130 > 100 + x$, then at least two numbers from $D_1, D_2, \dots, D_{64}, D_{65}, D_1 + x, D_2 + x, \dots, D_{64} + x, D_{65} + x$ must be same because pigeon hole represents quantity in this instance. However, if two values are to be same, $D_a = D_b + x$ because $D_a \neq D_b$ and $D_a + x \neq D_b + x$.

If $130 > 100 + x$, then $D_a = D_b + x$, or there always exists two days in which total of x problems are solved during those two days. Because $x < 30$, the maximum possible value for x is 29.

Conclusion: **29 is the maximum possible value of x .**

Problem 5

Key Word: Lucas's Theorem, Stars and Bars, Property of Modular

$\boxed{1246}$ 1246 Stars, 57 Bars $\frac{(1246+57)!}{1246!57!}$ $= 1303C_{57}$	$\boxed{896}$ 896 Stars, 57 Bars $\frac{(896+57)!}{896!57!}$ $= 953C_{57}$	$\boxed{4050}$ 4050 Stars, 57 Bars $\frac{(4050+57)!}{4050!57!}$ $= 4107C_{57}$	$\boxed{7547}$ 7547 Stars, 57 Bars $\frac{(7547+57)!}{7547!57!}$ $= 7604C_{57}$
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The total probability is $1303C_{57} \cdot 953C_{57} \cdot 4107C_{57} \cdot 7604C_{57}$

Applying Lucas's Theorem for r_7

$$\begin{aligned}
 57_{(10)} &= 111_{(7)} \\
 1303_{(10)} &= 3542_{(7)} \\
 953_{(10)} &= 2531_{(7)} \\
 4107_{(10)} &= 14655_{(7)} \\
 7604_{(10)} &= 31111_{(7)}
 \end{aligned}$$

Lucas's Theorem is applied to find r_7 .

$$\begin{aligned} {}_{1303}C_{57} &\equiv {}_3C_0 \cdot {}_5C_1 \cdot {}_4C_1 \cdot {}_2C_1 \pmod{7} \\ &\equiv 1 \cdot 5 \cdot 4 \cdot 2 \pmod{7} \\ &\equiv 40 \pmod{7} \\ &\equiv 5 \pmod{7} \end{aligned}$$

$$\begin{aligned} {}_{953}C_{57} &\equiv {}_2C_0 \cdot {}_5C_1 \cdot {}_3C_1 \cdot {}_1C_1 \pmod{7} \\ &\equiv 1 \cdot 5 \cdot 3 \cdot 1 \pmod{7} \\ &\equiv 15 \pmod{7} \\ &\equiv 1 \pmod{7} \end{aligned}$$

$$\begin{aligned} {}_{4107}C_{57} &\equiv {}_1C_0 \cdot {}_4C_0 \cdot {}_6C_1 \cdot {}_5C_1 \cdot {}_5C_1 \pmod{7} \\ &\equiv 1 \cdot 1 \cdot 6 \cdot 5 \cdot 5 \pmod{7} \\ &\equiv 150 \pmod{7} \\ &\equiv 3 \pmod{7} \end{aligned}$$

$$\begin{aligned} {}_{7604}C_{57} &\equiv {}_3C_0 \cdot {}_1C_0 \cdot {}_1C_1 \cdot {}_1C_1 \cdot {}_1C_1 \pmod{7} \\ &\equiv 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \pmod{7} \\ &\equiv 1 \pmod{7} \end{aligned}$$

$$r_7 \Rightarrow 5 \cdot 1 \cdot 3 \cdot 1 = 15 \Rightarrow 1 \quad (\because 15 \equiv 1 \pmod{7}) \quad \therefore r_7 = 1$$

Applying Lucas's Theorem for r_3

$$\begin{aligned} 57_{(10)} &= 2010_{(3)} \\ 1303_{(10)} &= 1201021_{(3)} \\ 953_{(10)} &= 1022022_{(3)} \\ 4107_{(10)} &= 111212100_{(3)} \\ 7604_{(10)} &= 101102122_{(3)} \end{aligned}$$

Lucas's Theorem is applied to find r_7 .

$$\begin{aligned} {}_{1303}C_{57} &\equiv {}_1C_0 \cdot {}_2C_0 \cdot {}_0C_0 \cdot {}_1C_2 \cdot {}_0C_0 \cdot {}_2C_1 \cdot {}_1C_0 \pmod{3} \\ &\equiv 1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 2 \cdot 1 \pmod{3} \\ &\equiv 0 \pmod{3} \end{aligned}$$

$$\begin{aligned} r_3 &= 0 \quad (\because {}_{1303}C_{57} \equiv 0 \pmod{3}) \\ \therefore r_3 &= 0 \end{aligned}$$

$$r_7 \cdot r_3 = 1 \cdot 0 = 0$$

Conclusion: $r_7 \cdot r_3$ is equal to zero.

Problem 10

Key Word: Cauchy-Schwarz Inequality

Part I

According to Cauchy-Schwarz inequality, for any number a_n and b_n , $(a_0^2 + \cdots + a_n^2)(b_0^2 + \cdots + b_n^2) \geq (a_0b_0 + \cdots + a_nb_n)^2$ is true.

$\frac{4}{3}x^2 + 5y^2 + \frac{1}{2}z^2 = 3$ and $f(x, y, z) = 3x + 2y - 6z$ are given by the problem. In order to find the maximum value for $f(x, y, z)$, Cauchy-Schwarz inequality can be utilized.

$$\left\{ \left(\sqrt{\frac{4}{3}}x \right)^2 + \left(\sqrt{5}y \right)^2 + \left(\sqrt{\frac{1}{2}}z \right)^2 \right\} \left\{ \left(\frac{3}{\sqrt{\frac{4}{3}}} \right)^2 + \left(\frac{2}{\sqrt{5}} \right)^2 + \left(\frac{-6}{\sqrt{\frac{1}{2}}} \right)^2 \right\} \geq (3x + 2y - 6z)^2$$

The Cauchy-Schwarz inequality was written to utilize $\frac{4}{3}x^2 + 5y^2 + \frac{1}{2}z^2 = 3$ and the fact that the maximum value of $3x + 2y - 6z$ must be found. The inequality may be simplified.

$$\left\{ \left(\frac{4}{3}x^2 \right) + (5y^2) + \left(\frac{1}{2}z^2 \right) \right\} \left\{ \left(\frac{9}{\frac{4}{3}} \right) + \left(\frac{4}{5} \right) + \left(\frac{36}{\frac{1}{2}} \right) \right\} \geq (3x + 2y - 6z)^2$$

Substituting $\frac{4}{3}x^2 + 5y^2 + \frac{1}{2}z^2$ provides the maximum value of $(3x + 2y - 6z)^2$.

$$\begin{aligned} \{3\} \left\{ \left(\frac{27}{4} \right) + \left(\frac{4}{5} \right) + (72) \right\} &\geq (3x + 2y - 6z)^2 \\ \{3\} \left\{ \left(\frac{135}{20} \right) + \left(\frac{16}{20} \right) + \left(\frac{1440}{20} \right) \right\} &\geq (3x + 2y - 6z)^2 \\ \{3\} \left\{ \frac{1591}{20} \right\} &\geq (3x + 2y - 6z)^2 \\ \frac{4773}{20} &\geq (3x + 2y - 6z)^2 \end{aligned}$$

By square rooting both sides, the maximum value of $3x + 2y - 6z$ could be found.

$$-\sqrt{\frac{4773}{20}} \leq \sqrt{(3x + 2y - 6z)^2} \leq \sqrt{\frac{4773}{20}}$$

$$-\sqrt{\frac{4773}{20}} \leq 3x + 2y - 6z \leq \sqrt{\frac{4773}{20}}$$

$$-\frac{1}{2}\sqrt{\frac{4773}{5}} \leq 3x + 2y - 6z \leq \frac{1}{2}\sqrt{\frac{4773}{5}}$$

Conclusion: **The maximum value of $f(x, y, z)$ is $\frac{1}{2}\sqrt{\frac{4773}{5}}$.**

Part II

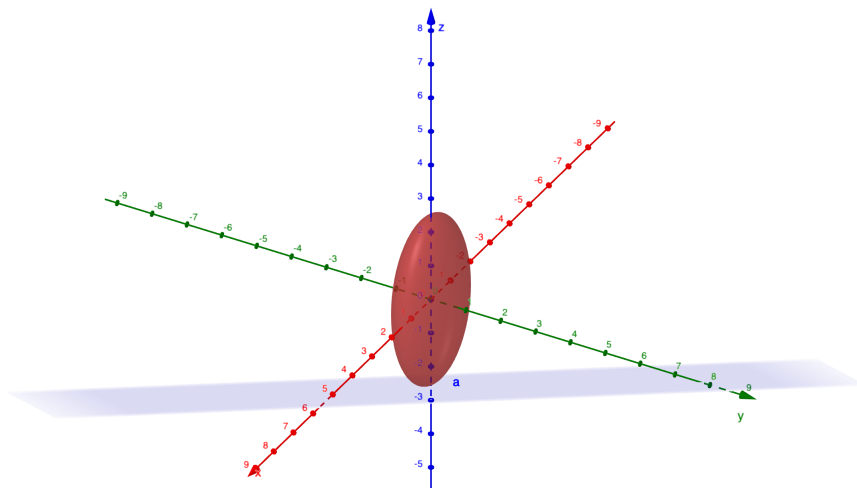


Figure 1: GeoGebra was utilized to manifest the current situation.

$3x + 2y - 6z = \sqrt{\frac{4773}{20}}$ is the blue plane that was established in Part I. By investigating the points where the plane meets $x - axis$, $y - axis$ and $z - axis$, the area formula for tetrahedron is available to discover the volume of the polyhedron enclosed by this plane and the constraints $x \geq 0$, $y \geq 0$, and $z \leq 0$.

Intersection with x -axis

$$y = 0, z = 0$$

$$3x + 2(0) - 6(0) = \sqrt{\frac{4773}{20}}$$

$$3x = \sqrt{\frac{4773}{20}}$$

$$\therefore x = \frac{1}{3}\sqrt{\frac{4773}{20}}$$

$(\frac{1}{3}\sqrt{\frac{4773}{20}}, 0, 0)$ is the intersection point.

Intersection with y -axis

$$x = 0, z = 0$$

$$3(0) + 2y - 6(0) = \sqrt{\frac{4773}{20}}$$

$$2y = \sqrt{\frac{4773}{20}}$$

$$\therefore y = \frac{1}{2}\sqrt{\frac{4773}{20}}$$

$(0, \frac{1}{2}\sqrt{\frac{4773}{20}}, 0)$ is the intersection point.

Intersection with z -axis

$$x = 0, y = 0$$

$$3(0) + 2(0) - 6z = \sqrt{\frac{4773}{20}}$$

$$-6z = \sqrt{\frac{4773}{20}}$$

$$\therefore z = -\frac{1}{6}\sqrt{\frac{4773}{20}}$$

$(0, 0, -\frac{1}{6}\sqrt{\frac{4773}{20}})$ is the intersection point.

Finding the Volume

$$\begin{aligned}
 \text{Volume} &= \frac{1}{3} \cdot \left| \frac{1}{3} \sqrt{\frac{4773}{20}} \right| \cdot \left| \frac{1}{2} \sqrt{\frac{4773}{20}} \right| \cdot \left| -\frac{1}{6} \sqrt{\frac{4773}{20}} \right| \\
 &= \frac{1}{3} \cdot \frac{1}{6} \cdot \frac{4773}{20} \cdot \frac{1}{6} \cdot \sqrt{\frac{4773}{5}} \\
 &= \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1591}{20} \cdot \frac{1}{6} \cdot \sqrt{\frac{4773}{5}} \\
 &= \frac{1591}{720} \cdot \frac{1}{2} \sqrt{\frac{4773}{5}} \\
 &= \frac{1591}{1440} \sqrt{\frac{4773}{5}} \\
 &= \frac{1591\sqrt{23865}}{7200}
 \end{aligned}$$

Conclusion: **The volume of the polyhedron enclosed by the plane and the constraints $x \geq 0$, $y \geq 0$ and $z \leq 0$ is $\frac{1591\sqrt{23865}}{7200}$.**

Conclusion: **The maximum value for $f(x, y, z)$ is $\frac{\sqrt{23865}}{10}$. The volume of the polyhedron enclosed by this plane and the constraints $x \geq 0$, $y \geq 0$ and $z \leq 0$ is $\frac{1591\sqrt{23865}}{7200}$.**

Problem 11

Key Word: Vieta's Theorem, AM-GM Inequality

The generalized form for AM-GM inequality states that $a_1 + a_2 + \dots + a_n \geq n \sqrt[n]{a_1 a_2 \dots a_n}$ for $a_n > 0$.

According to Vieta's theorem, the following equations are true.

$$\begin{aligned} -a &= r_1 + r_2 + \dots + r_8 \\ -b &= r_1 r_2 r_3 + \dots + r_6 r_7 r_8 \\ -c &= r_1 r_2 r_3 r_4 r_5 + \dots + r_4 r_5 r_6 r_7 r_8 \\ -d &= r_1 r_2 r_3 r_4 r_5 r_6 r_7 + \dots + r_2 r_3 r_4 r_5 r_6 r_7 r_8 \\ e &= r_1 r_2 r_3 r_4 r_5 r_6 r_7 r_8 \end{aligned}$$

The equations above are symmetric. Meaning, the following expression is true using AM-GM inequality.

$$-(a + b + c + d) = (r_1 + \dots + r_8) + (r_1 r_2 r_3 + \dots + r_6 r_7 r_8) + (r_1 \dots r_5 + \dots + r_4 \dots r_8) + (r_1 \dots r_7 + \dots + r_2 \dots r_8) \geq n \sqrt[n]{r_1 \dots r_8}^m$$

n , which represents the number of terms in the left side of the inequality, could be investigated using combination.

$$\begin{aligned} n &= {}_8C_1 + {}_8C_3 + {}_8C_5 + {}_8C_7 \\ &= 8 + 56 + 56 + 8 \\ &= 128 \end{aligned}$$

Using the symmetric nature of $-a$, $-b$, $-c$ and $-d$, it is evident that the number of r_1 equals the number of $r_1, r_2, r_3, r_4, r_5, r_6, r_7$ and r_8 . In another terms, m is equal to the number of r_1 . m could be found by assuming that r_1 is already chosen and choosing either 0, 2, 4 or 6 terms from r_2, r_3, r_4, r_5, r_6 and r_7 because 1, 3, 5 or 7 terms are selected in total from $r_1, r_2, r_3, r_4, r_5, r_6, r_7$ and r_8 .

$$\begin{aligned} m &= {}_7C_0 + {}_7C_2 + {}_7C_4 + {}_7C_6 \\ &= 1 + 21 + 35 + 7 \\ &= 64 \end{aligned}$$

The following inequality may be written by substituting the known values.

$$-(a + b + c + d) \geq 128 \sqrt[128]{e^{64}}$$

Because both $-(a + b + c + d)e$ and $128 \sqrt[128]{e^{64}}$ are positive numbers, the entire inequality continues to satisfy when both sides are squared.

$$(-(a + b + c + d))^2 \geq \left(128 \sqrt[128]{e^{64}}\right)^2$$

$$(a + b + c + d)^2 \geq \left(128e^{\frac{1}{2}}\right)^2$$

$$(a + b + c + d)^2 \geq 2^{14}e$$

$$\frac{(a + b + c + d)^2}{e} \geq 2^{14}$$

The least possible value of $\frac{(a+b+c+d)^2}{e}$ is 2^{14} , or 16384.

Conclusion: The minimum possible value for $\frac{(a+b+c+d)^2}{e}$ is 16384.

Problem 14

Key Word: Mass Point

The diagram shows a triangle with vertices A (top), B (bottom-left), and C (bottom-right). The side lengths are AB = 11, BC = 10, and AC = 10. Mass points are assigned to the vertices and points on the sides:

- Vertex A: mass 110
- Vertex B: mass 30
- Vertex C: mass 15
- Point D on BC: mass 9
- Point E on AC: mass 20
- Point F on AB: mass 6

The centroid P is the intersection of the medians AD, BE, and CF. The following calculations are shown:

$$\frac{AP}{PD} = \frac{65}{20} = \frac{655}{440} = \frac{5}{4}$$

$$\frac{EP}{PF} = \frac{6}{20} = \frac{3}{10}$$

$$\frac{5}{9} + \frac{3}{10} = \frac{50+27}{90} = \frac{77}{90}$$