Desargues' Theorem

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Contents

1 Prerequisite Knowledge

Desargues' Theorem is a renowned theorem that relates the axis and the center of perspective of two triangles. Therefore, before we dive into Desargues' Theorem, we need to have a gut feeling on Projective Geometry. As I mentioned, because we only need a "gut feeling", it is not necessary to attain profound insights on Projective Geometry.

First and foremost, consider the two, not necessarily similar, triangles and a point P.

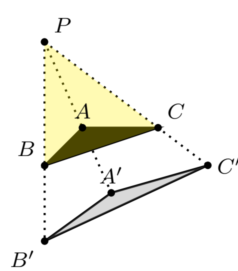


Figure 1

Let's try and think of it like this. We have a light source P that shines $\triangle ABC$. There, we get our outcome $\triangle A'B'C'$. Henceforth, we say that A', B', C' are corresponding points of A, B, C respectively. Similarly, we could state that $\overline{A'B'}, \overline{B'C'}, \overline{C'A'}$ are corresponding sides of $\overline{AB}, \overline{BC}, \overline{CA}$ respectively.

2 Desargues' Theorem

Theorem 2.1: Desargues' Theorem

Two triangles are axially perspective *iff* they are centrally perspective. Two triangles are considered axially perspective if there exists the axis of perspectivity and they are centrally perspective if there exists the center of perspectivity [1].

For simpler explanation, consider the diagram below, which is simply the extension of Figure 1.

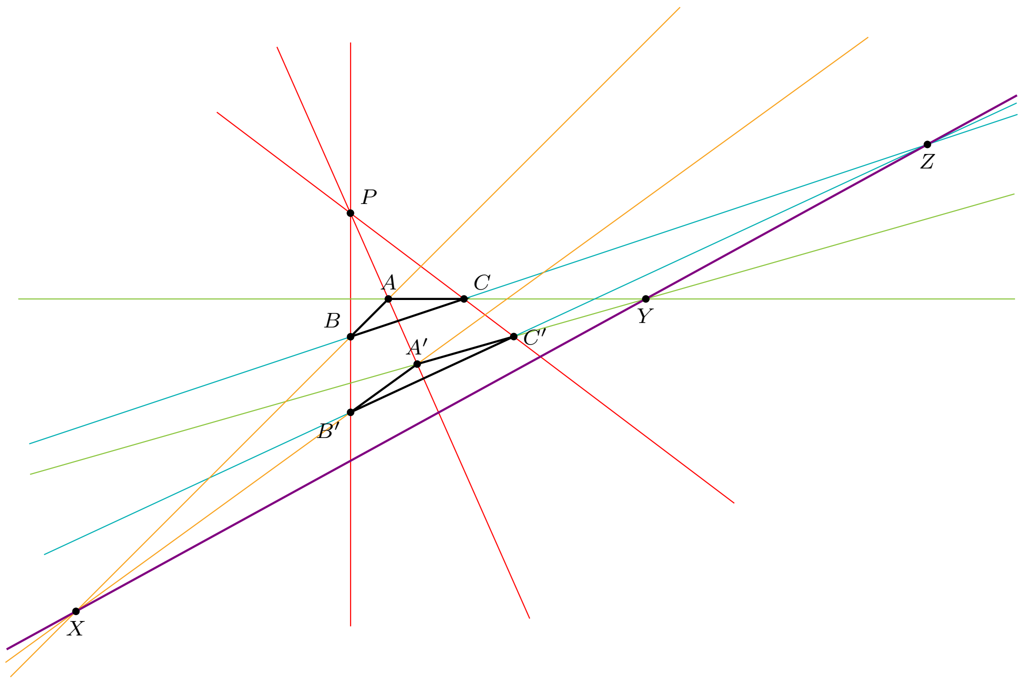


Figure 2

Point P, where AA', BB', CC' concur, is the center of perspectivity. Line XZ, where X, Y, Z are collinear, is the axis of perspectivity in the diagram above. If the pairs of corresponding lines are all parallel, it is said that the lines meet at a point of infinity. However, the case is not commonly discussed.

The theorem is stating that if there exists a point P such that $P = AA' \cap BB' \cap CC'$, then the points $X = AB \cap A'B'$, $Y = BC \cap B'C'$, and $Z = CA \cap C'A'$ are collinear.

3 Proof of Desargues' Theorem

Theorem 3.1 (Desargues' Theorem)**.** *Two triangles are axially perspective* iff *they are centrally perspective.*

Proof. The proof involves the use of Menelaus' theorem three times [2]. Recall that the Menelaus' Theorem could be defined as the following.

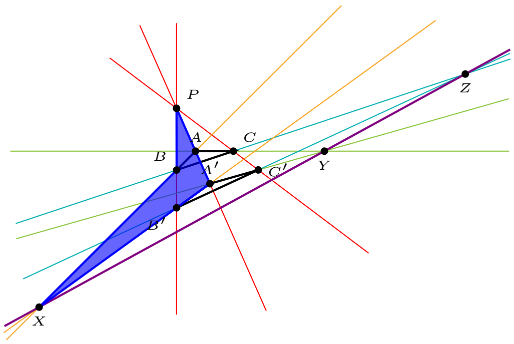
Theorem 3.1: Menelaus' Theorem

Assume for $\triangle ABC$, X, Y, Z are defined as $X \in \overline{AB}$, $Y \in \overline{BC}$, and $Z \in \overline{CA}$. Then the following equation of cross-ratio holds.

$$\frac{XA}{XB} \cdot \frac{YB}{YC} \cdot \frac{ZC}{ZA} = -1$$

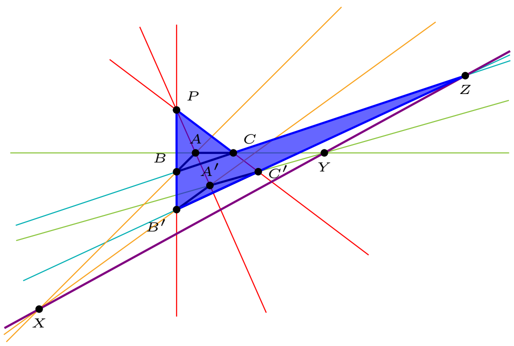
Because Desargues' Theorem must satisfy the if and only if condition, let's first prove that two triangles are axially perspective if they are centrally perspective.

Assuming that $\triangle ABC$ and $\triangle A'B'C'$ are perspective from point P, Menelaus' Theorem could be used.



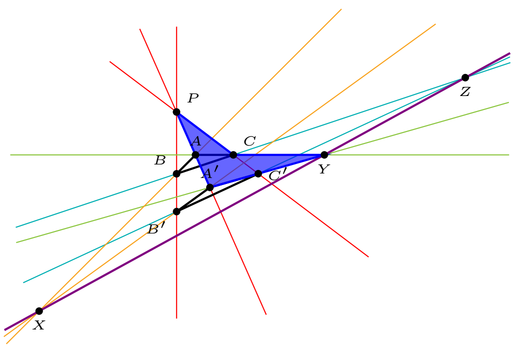
Consider $\triangle PA'B'$ with transverse line XB. Using Menelaus' Theorem, the following equation could be derived.

$$\frac{XA'}{XB'} \cdot \frac{BB'}{BP} \cdot \frac{AP}{AA'} = 1$$



Similarly, consider $\triangle PB'C'$ and transverse line ZC.

$$\frac{ZB'}{ZC'} \cdot \frac{CC'}{CP} \cdot \frac{BP}{BB'} = 1$$



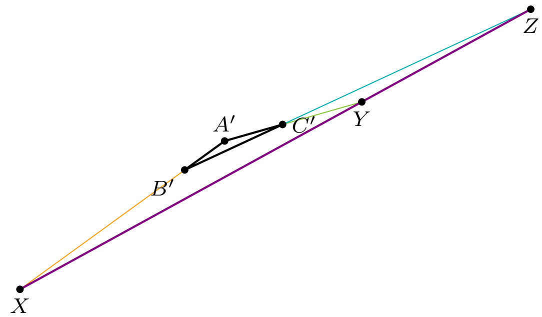
With $\triangle PC'A'$ and line YC,

$$\frac{YA'}{YC'} \cdot \frac{CC'}{CP} \cdot \frac{AP}{AA'} = 1$$
$$\frac{YC'}{YA'} \cdot \frac{CP}{CC'} \cdot \frac{AA'}{AP} = 1$$

Let's multiply our equations.

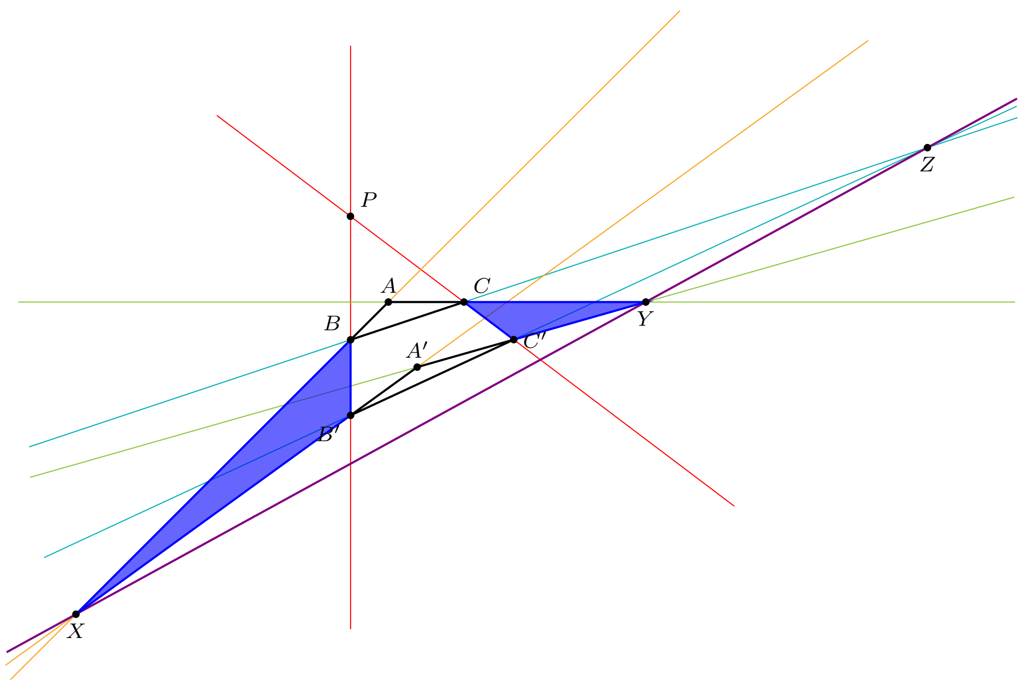
$$\left(\frac{XA'}{XB'} \cdot \frac{BB'}{BP} \cdot \frac{AP}{AA'}\right) \cdot \left(\frac{ZB'}{ZC'} \cdot \frac{CC'}{CP} \cdot \frac{BP}{BB'}\right) \cdot \left(\frac{YC'}{YA'} \cdot \frac{CP}{CC'} \cdot \frac{AA'}{AP}\right) = 1$$
$$\frac{XA'}{XB'} \cdot \frac{ZB'}{ZC'} \cdot \frac{YC'}{YA'} = 1$$

With the equation, consider the diagram below.



By converse of Menelaus' Theorem, it is evident that X, Y, and Z are collinear.

Now that the first statement for if and only if condition is proven, the reverse must also be proven. Meaning, the statement "two triangles are centrally perspective if they are axially perspective" must be proven. Consider the diagram below. The same construction method was used. However, the if AA', BB', and CC' concur is yet to be known.



Notice that $\triangle XBB'$ and $\triangle YCC'$ are in perspective from point Z. Moreover, the first half of the proof manifested that P, A, and A' must be collinear. By construction, P is on line BB' and CC'. Moreover, it is also on the line AA'. Thus, $\triangle ABC$ and $\triangle A'B'C'$ are centrally perspective if they are axially perspective. □

4 Problem

Problem 4.1: Problem

D is a point on $\overline{BC}$ in $\triangle ABC$. Let I_1, I_2 be the incenter of $\triangle ABD$ and $\triangle ACD$ respectively. Moreover, let I_3 and I_4 be ex-centers in respect to $\angle BAD$ and $\angle CAD$ respectively. Show that $\overline{I_1I_2}$, $\overline{I_3I_4}$, and $\overline{BC}$ intersect at one point. (Source: Unknown)

Proof. First, because I_1 and I_2 are the angle bisectors of $\angle ABD$ and $\angle ACD$ respectively, $I = BI_1 \cap CI_2$ is the incenter of $\triangle ABC$. Similarly, we know that $I' = BI_3 \cap CI_4$ is the ex-center of $\triangle ABC$ in respect that $\angle BAC$. Therefore A, I, and I' are collinear.

With similar idea, we could prove that A, I_1, I_3 are collinear as well as A, I_2, I_4. In other words, I_1I_3 and I_2I_4 intersect at point A. Therefore, $BI_1 \cap CI_2$, $I_1I_3 \cap I_2I_4$, and $I_3B \cap I_4C$ are collinear in $\triangle I_1BI_3$ and $\triangle I_2CI_4$. By Desargues' theorem, $\overline{I_1I_2}$, $\overline{I_3I_4}$, and $\overline{BC}$ concurs. □

You can check out much more detailed solution here: Solution :L)

References

[1] Alexander Katz, Henry Maltby, Sal Gard, and et.al. Projective geometry. `https://brilliant.org/wiki/projective-geometry/`, n.d. Accessed: 2025-08-15.

[2] Paris Pamfilos. Desargues' theorem and perspectivities. `https://users.math.uoc.gr/~pamfilos/eGallery/problems/Desargues.pdf`, 2024. Accessed: 2025-08-15.