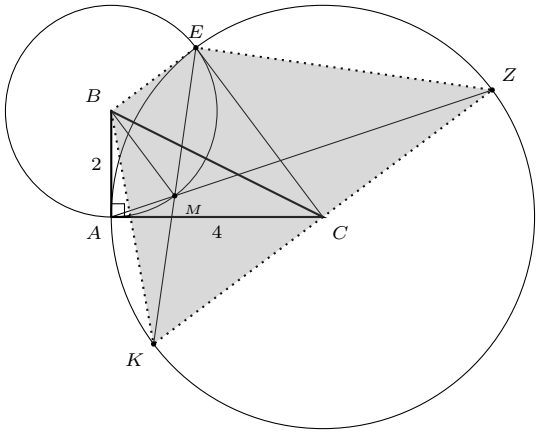
2017 PUMaC Geometry A Problem 6

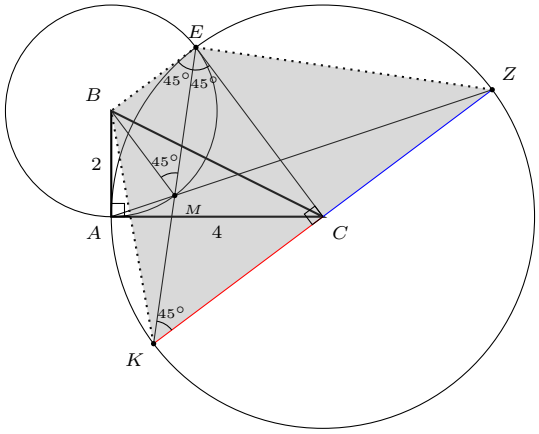
Triangle ABC has $\angle A = 90°$, $AB = 2$, and $AC = 4$. Circle ω_1 has center C and radius CA, while circle ω_2 has center B and radius BA. The two circles intersect at point E, different from point A. Point M is on ω_2 and in the interior of ABC, such that BM is parallel to EC. Suppose EM intersects ω_1 at point K and AM intersects ω_1 at point Z. What is the area of quadrilateral $ZEBK$?

Video explanation for anyone interested :))

Solution Consider the diagram below.



First, notice that $\angle BEC = 90°$ because $\triangle ABC \cong \triangle EBC$ by SSS congruence. Moreover, $\triangle BME$ is an isosceles triangle with $BM = BE = 2$. Furthermore, since $BM \parallel EC$ by construction, $\angle BME = \angle MEC$. Therefore, $\angle BEM = \angle BME = \angle MEC = \angle MKC = 45°$, for $\angle BEM + \angle MEC = 90°$.

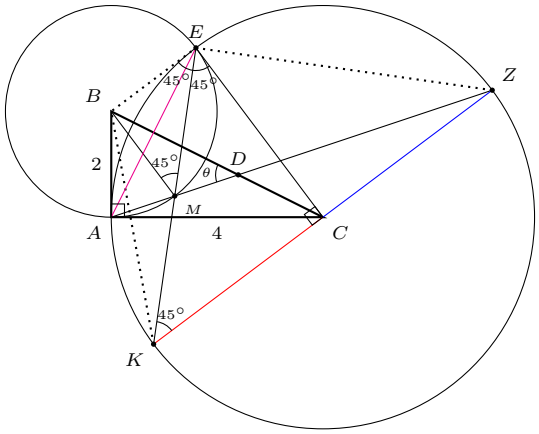


Considering the diagram, if points K, C, Z are collinear, the area of the quadrilateral $ZEBK$ could be investigated readily.

Lemma. *Points K, C, Z are collinear.*

Proof. The fact that $\angle KCE = 90°$ was demonstrated. Therefore, it suffices to show that $\angle ECZ = 90°$.

Notice that $\angle AZC = \angle CAZ$ and $\angle EZA = \frac{1}{2}\angle ECA = \angle BCA$. Therefore, $\angle EZC = \angle BDA$ from the diagram below.



Because AE is the radical axis of ω_1 and ω_2, $AE \perp BC$.

$$\angle EAD = \frac{1}{2}\angle EBM = 45^\circ$$

Thus, $\theta = 45^\circ = \angle EZK$. Because $\triangle ECZ$ is an isosceles triangle, $\angle ECZ = 90^\circ$. □

The proven lemma could be utilized to compute the area.

$$\begin{aligned}
[BEZK] &= [\triangle BKE] + [\triangle EKZ] \\
&= \frac{1}{2} \cdot BE \cdot KE \cdot \sin 45^\circ + \frac{1}{2} \cdot KZ \cdot CE \\
&= \frac{1}{2} \cdot 2 \cdot 4\sqrt{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot 8 \cdot 4 \\
&= 4 + 16 \\
&= 20
\end{aligned}$$

Thus, $[BEZK] = 20$. □