

Problem

A , B , C , and D stand in a circle. Simultaneously, each player selects another player at random and points at that person, who must then sit down. What is the probability that A is the only person who remains standing?

Solution

Key Word Counting Strategy

Solution I

total cases of only A standing = total cases of A standing – total cases of A and (B , C , or D) standing

$$\begin{aligned}
 &= \frac{3}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} - \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot {}_3C_2 \\
 &= \frac{8}{27} - \frac{4}{27} \\
 &= \boxed{\frac{4}{27}}
 \end{aligned}$$

Solution II

$$\begin{array}{cccc}
 A & B & C & D \\
 \hline
 B & C & D & B \\
 & & & C \\
 & & D & \\
 & C & & \\
 & D & &
 \end{array}$$

$$\therefore \text{total cases of only } A \text{ standing} = \frac{3 \cdot 2 \cdot 1 \cdot 2}{3^4} = \boxed{\frac{4}{27}}$$

□

Problem

R is blindfolded and standing 1 step away from an ice cream stand. Every second, he has a $\frac{1}{4}$ probability of walking 1 step towards the ice cream stand, and a $\frac{3}{4}$ probability of walking 1 step away from the ice cream stand. When he is 0 steps away from the ice cream stand, he wins. What is the probability that R eventually wins?

Solution

Key Word Recursion

First and foremost, variables for recursion formula could be set.

$$\begin{aligned} S_1 &= \text{Probability to win from 1} \\ S_2 &= \text{Probability to win from 2} \\ &\vdots \end{aligned}$$

Using the defined variables, recursion equations could be set.

$$\begin{aligned} S_1 &= \frac{1}{4} + \frac{3}{4}S_2 \\ S_2 &= \frac{1}{4}S_1 + \frac{3}{4}S_3 \\ S_3 &= \frac{1}{4}S_2 + \frac{3}{4}S_4 \\ &\vdots \\ S_n &= \frac{1}{4}S_{n-1} + \frac{3}{4}S_{n+1} \end{aligned}$$

Notice that S_1 is the value that need to be computed since R is located at 1.

$$S_1 + S_2 + \cdots + S_n = \frac{1}{4} + \frac{1}{4}(S_1 + S_2 + \cdots + S_{n-1}) + \frac{3}{4}(S_2 + \cdots + S_{n+1})$$

For simplicity, let $A = S_1 + S_2 + \cdots + S_n$.

$$\begin{aligned} A &= \frac{1}{4} + \frac{1}{4}(A - S_n) + \frac{3}{4}(A - S_1 + S_{n+1}) \\ 0 &= 1 - S_n - 3S_1 + 3S_{n+1} \end{aligned}$$

However, notice that $\lim_{n \rightarrow \infty} S_n = 0$.

$$\therefore 0 = 1 - 3S_1$$

$$S_1 = \boxed{\frac{1}{3}}$$

□

Problem

Enoch randomly hops between 5 leaves, on each turn hopping to one of the other 4 leaves with equal probability. After 5 hops, what is the probability that Enoch has returned to the leaf where he started?

Solution

Key Word Recursion

First and foremost, variables must be set in order to use recursion.

$$h_x = \text{Probability to be on the leaf where he started after } x \text{ hops}$$

$$h_{x-1} = \frac{1}{4}(1 - h_{x-1})$$

Using the formula above, substitution may lead to the correct answer.

$$h_5 = \frac{1}{4}(1 - h_4) = \boxed{\frac{51}{256}}$$

$$h_4 = \frac{1}{4}(1 - h_3) = \frac{13}{64}$$

$$h_3 = \frac{1}{4}(1 - h_2) = \frac{3}{16}$$

$$h_2 = \frac{1}{4}(1 - h_1) = \frac{1}{4}$$

$$h_1 = 0$$

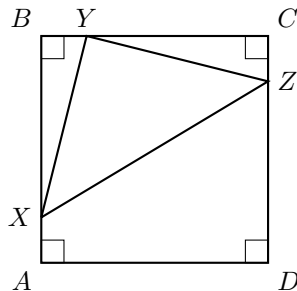
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Problem

Let $ABCD$ be a unit square and let X, Y, Z be points on sides AB, BC, CD , respectively, such that $AX = BY = CZ$. If the area of triangle XYZ is $\frac{1}{3}$, what is the maximum value of the ratio $\frac{XB}{AX}$?

Solution

Key Word Area Formula



Let $AX = BY = CZ = a$.

$$\begin{aligned}
 1 - 2 \cdot \frac{a(1-a)}{2} - \frac{((1-a)+a) \cdot 1}{2} &= \frac{1}{3} \\
 1 - a(1-a) - \frac{1}{2} &= \frac{1}{3} \\
 a &= \frac{3 \pm \sqrt{3}}{6}
 \end{aligned}$$

The maximum value of $\frac{1}{a} - 1$ must be computed

$$\therefore \max \left(\frac{1}{a} - 1 \right) = \boxed{2 + \sqrt{3}}$$

□