2021 AMC 12B Problem 18

Let z be a complex number satisfying $12|z|^2 = 2|z+2|^2 + |z^2+1|^2 + 31$. What is the value of $z + \frac{6}{z}$?

(A) -2 **(B)** -1 **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 4

Solution

Key Word Completing Squares

Let $z = a + bi$. Substitution of z could be done.

$$\begin{aligned} 12(a^2+b^2) &= 2\{(a+2)^2+b^2\} + \{(a^2-b^2+1)^2 + (2ab)^2\} + 31 \\ 12a^2 + 12b^2 &= 2a^2 + 8a + 8 + 2b^2 + a^4 + b^4 + 1 - 2a^2b^2 + 2a^2 - 2b^2 + 4a^2b^2 + 31 \\ a^4 + b^4 - 8a^2 - 12b^2 + 2a^2b^2 + 8a + 40 &= 0 \\ (a^2+b^2)^2 - 12(a^2+b^2) + 4(a^2+2a+1) + 36 &= 0 \\ (a^2+b^2-6)^2 + 4(a+1)^2 &= 0 \end{aligned}$$

$$\therefore\ a^2+b^2-6=0,\ a=-1,\ b=\sqrt{5}$$

$$z + \frac{6}{z} = \frac{a^2-b^2+2abi+6}{a+bi} = \frac{2-2\sqrt{5}i}{-1+\sqrt{5}i} = \boxed{\textbf{(A)}\ -2}$$

□

2021 AMC 12B Problem 20

Let $Q(z)$ and $R(z)$ be the unique polynomials such that

$$z^{2021} + 1 = (z^2+z+1)Q(z) + R(z)$$

and the degree of R is less than 2. What is $R(z)$?

(A) $-z$ **(B)** -1 **(C)** 2021 **(D)** $z+1$ **(E)** $2z+1$

Solution

Key Word $x^3 - 1 = (x-1)(x^2+x+1)$

GENERAL RULE OF THUMB: When you see x^2+x+1, you should immediately think of $x^3-1 = (x-1)(x^2+x+1)$. Let r_1, r_2, r_3 be the roots of $z^3 - 1 = 0$. WLOG, let $r_1 \neq 1$.

$$\begin{aligned} {r_1}^{2021} + 1 &= 0 + R(r_1) \\ r^2 + 1 &= R(r_1) \\ R(r_1) &= -r_1\ (\because {r_1}^2 + r_1 + 1 = 0) \\ &\Rightarrow \boxed{\textbf{(A)}\ -z} \end{aligned}$$

□