

Problem

Four couples are to be seated in a row. If it is required that each woman may only sit next to her husband or another woman, how many different possible seating arrangements are there?

Solution

Key Word Counting Strategy

i) One Pair Together

w w w w m m m m

$${}_4C_1 \cdot 3! \cdot 3! \cdot 2 = \underline{288}$$

ii) Two Pairs Together

w w w m m m m w

w w m m m m w w

w m m m m w w w

$$({}_4C_2 \cdot 2 \cdot 2 \cdot 2) \cdot 2 \cdot 3 = \underline{288}$$

iii) Three Pairs Together

w w m m w w m m

w m m w w w m m

w w m m m w w m

w m m m w w w m

$${}_4C_3 \cdot 2 \cdot 3! \cdot 4 = \underline{192}$$

iiii) Four Pairs Together

w m m w w m m w

$$4! \cdot 2 = \underline{48}$$

$$288 + 288 + 192 + 48 = \boxed{816}$$

□

Problem

How many ten-digit positive integers consist of ten different digits and are divisible by 99?

Solution

Key Word Counting Strategy

Because the sum of all integers from 0 to 9 is 45, only the divisibility rule for 11 must be checked. The pairs of possible sums of alternative digits are (17, 28) and (6, 39). However, because the case (6, 39) is not possible, only the pair (17, 28) must be checked. Because counting for 28 seems easier, let's try counting for 28.

9	8	6	5	0
9	8	6	4	1
9	8	6	3	2
9	8	5	4	2
9	7	6	5	1
9	7	6	4	2
9	7	5	4	3
8	7	6	5	2
8	7	6	4	3

$$\therefore (5! - 4!)5! + 8 \cdot 5!5! + 5!5! + 8(5! - 4!)5! = \boxed{233280}.$$

□

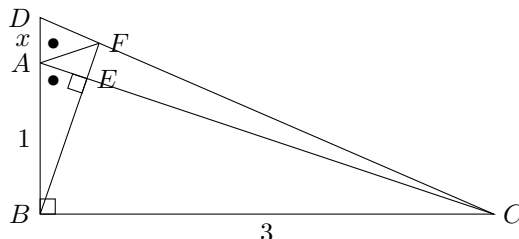
Problem

$\triangle ABC$ is right-angled at B , with $AB = 1$ and $BC = 3$. E is the foot of perpendicular from B to AC . BA and BE are produced to D and F respectively such that D, F, C are collinear and $\angle DAF = \angle BAC$. Find the length of AD .

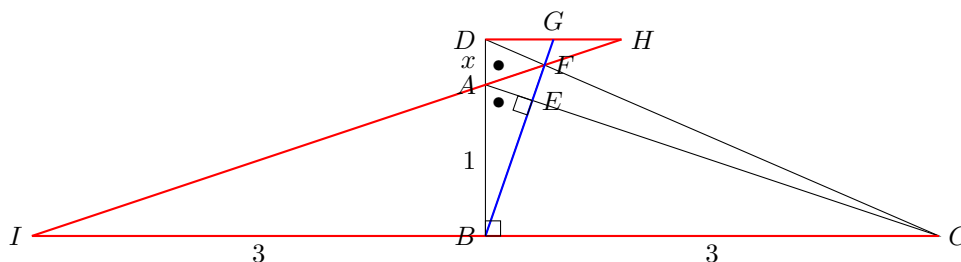
Solution

Key Word Similar Triangle

First and foremost, let us draw the situation first.



An impulse to extend BF is created. Different extension lines could be drawn to use similar triangles.



Looking carefully at the diagram, it could be inferred that $DG = GH$. The reason is because $\triangle DFS \sim \triangle CFI$ and $IB = BC$. In another words, $DG = \frac{3x}{2}$. Using the similar triangles again, $\frac{3x}{2} : x + 1 = 1 : 3$ is

true. Therefore, $x = \boxed{\frac{2}{7}}$. □

2021 AMC 12A Problem 12

All the roots of the polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of B ?

- (A) -88 (B) -80 (C) -64 (D) -41 (E) -40

Solution

Key Word Vieta's Formula

Using Vieta's formula, it is true that $r_1 + r_2 + r_3 + r_4 + r_5 + r_6 = 10$ and $r_1 r_2 r_3 r_4 r_5 r_6 = 16$. Moreover, $r_1 r_2 r_3 + \dots + r_4 r_5 r_6$ is the value that needs to be computed. Remember.. all roots are positive integers!! Thus, WLOG, $r_1 = 1, r_2 = 1, r_3 = 2r_4 = 2, r_5 = 2$ and $r_6 = 2$.

Now, let's be smart. The number of cases of choosing 2,2,2 is ${}_4C_3$. Moreover, the number of cases of selecting 2,2,1 is ${}_4C_2 \cdot {}_2C_1$. Furthermore, the number of cases of choosing 2,1,1 is ${}_4C_1 \cdot {}_2C_2$. Therefore, $8 \cdot {}_4C_3 + 4 \cdot {}_4C_2 \cdot {}_2C_1 + 2 \cdot {}_4C_1 \cdot {}_2C_2 = 32 + 48 + 8 = 88$. We now need to change the sign, since we used Vieta's Formula. Thus, the answer is $\boxed{\text{(A)} -88}$. □

2021 AMC 12A Problem 22

Suppose that the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos \frac{2\pi}{7}$, $\cos \frac{4\pi}{7}$, and $\cos \frac{6\pi}{7}$, where angles are in radians. What is abc ?

- (A) $-\frac{3}{49}$ (B) $-\frac{1}{28}$ (C) $\frac{\sqrt[3]{7}}{64}$ (D) $\frac{1}{32}$ (E) $\frac{1}{28}$

Solution

Key Word Vieta's Formula, The Roots of Unity, Euler's Formula, Trigonometric Identities

First and foremost, using the Vieta's Formula, it is evident that

$$\begin{aligned} -a &= \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} \\ b &= \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} + \cos \frac{2\pi}{7} \cos \frac{6\pi}{7} + \cos \frac{4\pi}{7} \cos \frac{6\pi}{7} \\ -c &= \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} \cos \frac{6\pi}{7} \end{aligned}$$

Calculating a

Using the 7th roots of unity, we could know that

$$e^{2\pi \cdot \frac{0}{7}} + e^{2\pi \cdot \frac{1}{7}} + \dots + e^{2\pi \cdot \frac{5}{7}} + e^{2\pi \cdot \frac{6}{7}}$$

is true. Moreover, using the Euler's Formula $e^{i\theta} = \cos \theta + i \sin \theta$, it is evident that the following is true.

$$\cos \left(2\pi \cdot \frac{0}{7} \right) + \dots + \cos \left(2\pi \cdot \frac{6}{7} \right) + i \left(\sin \left(2\pi \cdot \frac{0}{7} \right) + \dots + \sin \left(2\pi \cdot \frac{6}{7} \right) \right) = 0$$

Using the trigonometric identities, the following equation may be modified.

$$1 + 2 \left(\cos \left(2\pi \cdot \frac{1}{7} \right) + \cos \left(2\pi \cdot \frac{2}{7} \right) + \cos \left(2\pi \cdot \frac{3}{7} \right) \right) = 0$$

In another words, $2(-a) = -1$ and $a = \frac{1}{2}$.

Calculating b

The trigonometric identity $\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta))$ could be used.

$$\begin{aligned} b &= \frac{1}{2} \left(\cos \frac{6\pi}{7} + \cos \frac{2\pi}{7} \right) + \frac{1}{2} \left(\cos \frac{8\pi}{7} + \cos \frac{4\pi}{7} \right) + \frac{1}{2} \left(\cos \frac{10\pi}{7} + \cos \frac{2\pi}{7} \right) \\ &= \frac{1}{2} \left(2 \cos \frac{2\pi}{7} + \cos \frac{6\pi}{7} + \cos \frac{8\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{10\pi}{7} \right) \\ &= \cos \frac{2\pi}{7} + \cos \frac{6\pi}{7} + \cos \frac{4\pi}{7} \\ &= -a \\ &= -\frac{1}{2} \end{aligned}$$

Calculating c

$$\begin{aligned}
 -c &= \frac{1}{2} \left(\cos \frac{6\pi}{7} + \cos \frac{2\pi}{7} \right) \left(\cos \frac{6\pi}{7} \right) = \left(\cos \frac{6\pi}{7} + \cos \frac{2\pi}{7} + \cos \frac{6\pi}{7} \right) \\
 &= \frac{1}{2} \left(\frac{1}{2} \left(\cos \frac{12\pi}{7} + \cos \frac{0\pi}{7} \right) + \frac{1}{2} \left(\cos \frac{8\pi}{7} + \cos \frac{4\pi}{7} \right) \right) \\
 &= \frac{1}{4} \left(1 + \cos \frac{12\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{8\pi}{7} \right) \\
 &= \frac{1}{4} (1 - a) \\
 &= \frac{1}{8}
 \end{aligned}$$

Final Calculation

$$abc = \frac{1}{2} \left(-\frac{1}{2} \right) \left(-\frac{1}{8} \right) = \boxed{\textbf{(D)} \quad \frac{1}{32}}$$

□