2022 AMC 12A Problem 16

A triangular number is a positive integer that can be expressed in the form $t_n = 1 + 2 + 3 + \cdots + n$, for some positive integer n. The three smallest triangular numbers that are also perfect squares are $t_1 = 1 = 1^2, t_8 = 36 = 6^2$, and $t_{49} = 1225 = 35^2$. What is the sum of the digits of the fourth smallest triangular number that is also a perfect square?

(A) 6 **(B)** 9 **(C)** 12 **(D)** 18 **(E)** 27

Solution

Key Word Pell's Equation

For arbitrary integers k, pairs of (n, k) that satisfy $\frac{n(n+1)}{2} = k^2$ must be found. The equation could be rewritten as $n^2 + n - 2k^2 = 0$. Because n and k are positive integers, an impulse to find an cleaner relationship for trial and error is created. $(n + \frac{1}{2})^2 - 2k^2 = \frac{1}{4}$. Multipying both sides by 4 may provide a better equation.

$$(2n + 1)^2 - 2(2k)^2 = 1$$

Let $x = 2n + 1$ and $y = 2k$ for cleaner view. $x^2 - 2y^2 = 1$ seems familiar. It is a specific Pell's Equation!!

The fundamental solution is given by the problem which is $(2 + 1, 2 \cdot 1)$, or $(3, 2)$. Because the general solution could be represented as $(x_i, y_i) = (x_1 + y_1\sqrt{D})^i_{(i\geq 1)}$, substitution is required.

$$(x_i, y_i) = (3 + 2\sqrt{2})^i$$

$$\begin{aligned} &\text{For } i = 4, \\ (x_4, y_4) &= (3 + 2\sqrt{2})^4 \\ &= (17 + 12\sqrt{2})^2 \\ &= 577 + 408\sqrt{2} \end{aligned}$$

Therefore, $x_4 = 577$ and $y_4 = 408$. In another words, $n_4 = 288$ and $k_4 = 204$. Because $204^2 = 41616$, $4 + 1 + 6 + 1 + 6 = \boxed{\textbf{(D) } 18}$. $\square$

2022 AMC 12A Problem 20

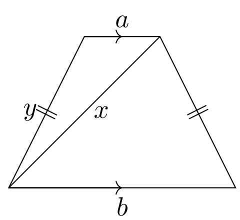
Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1, PB = 2, PC = 3$, and $PD = 4$. What is $\frac{BC}{AD}$?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

Solution

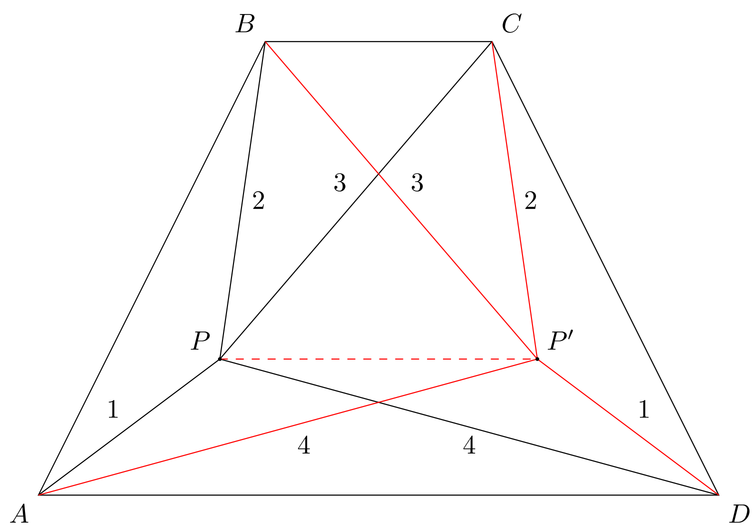
Key Property

Property I: Derived from Ptolemy's Theorem



$$ab = x^2 - y^2$$

General Rule of Thumb: When an isosceles trapezoid is given, draw a symmetric pair!!



Using the property of isosceles trapezoid, it is evident that $BC \cdot PP'- = 3^2 - 2^2 = 5$. Moreover, $PP' \cdot AD = 4^2 - 1^1 = 15$. Therefore, $\frac{BC \cdot PP'}{PP' \cdot AD} = \frac{BC}{AD} = \frac{5}{15} = \boxed{\textbf{(B)}\ \frac{1}{3}}$. $\square$