2020 KMO Problem 2.

Enoch Yu

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $x^2 f(x) + y f(y^2) = f(x+y) f(x^2 - xy + y^2)$ for all $x, y \in \mathbb{R}$.

1. $f(x) = x$

2. $x = y \neq 0$

$0 + 0 = f(0)^2$

$\therefore f(0) = 0$

3. $x \neq 0$, $y = x$

$x f(x^2) = f(x) f(x^2)$

$\Rightarrow$ If $f(x^2) \neq 0$, then $f(x) = x$

4. $y = -x$

$x^2 f(x) - x f(x^2) = 0$

$\Rightarrow x f(x) = f(x^2)$

If $x f(x) \neq 0$, then $f(x) = x$

$f(x) = 0$ or $f(x) = x$

$f(\alpha) = 0$ $(\alpha \neq 0)$, $f(\beta) = \beta$ $(\beta \neq 0, \alpha \neq \beta)$

$\alpha^2 f(\alpha) + \beta f(\beta^2) = f(\alpha+\beta) f(\alpha^2 - \alpha\beta + \beta^2)$

$\hookrightarrow \beta f(\beta)$

$\beta^3 = f(\alpha+\beta) f(\alpha^2 - \alpha\beta + \beta^2) \neq 0 \iff f(\alpha+\beta) \neq 0$

$= (\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)$ $\qquad f(\alpha^2 - \alpha\beta + \beta^2) \neq 0$

$\therefore \alpha^3 = 0 \iff \alpha = 0$

$\iff f(x+y) = x+y$

$\iff f(x^2 - xy + y^2) = x^2 - xy + y^2$

$f(x) = x$

$f(x) \equiv 0$

□